New Developments in Projective Geometric Algebra

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About the Speaker

- Working in game/graphics dev since 1994
- Previously at Sierra, Apple, Naughty Dog

Current projects:
- Slug Library, C4 Engine, The 31st, FGED, OpenGEX
More Information

- projectivegeometricalgebra.org
- Past GDC sessions on Grassmann algebra
- Foundations of Game Engine Development, Volume 1: Mathematics
Outline

- Take a look at conventional math
  - Pieces of a puzzle, but big picture missing

- Review of Grassmann algebra
  - With some new stuff added

- New developments in geometric algebra
  - Antiproducts, geometric norms, motors, flectors
Homogeneous Coordinates

- Add $w$ coordinate to make 4D vector
- Points have $w \neq 0$
- Directions have $w = 0$
- Allows rotation and translation to be combined in a single $4 \times 4$ matrix:

$$p' = \begin{bmatrix} R_{00} & R_{01} & R_{02} & T_x \\ R_{10} & R_{11} & R_{12} & T_y \\ R_{20} & R_{21} & R_{22} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$
Homogeneous Coordinates

- “Homogeneous” means any scalar multiple of a vector has the same geometric meaning
- Project into 3D space by intersecting with the plane $w = 1$
Implicit Planes

- Four-component quantity \((n_x, n_y, n_z, d)\)
- \(\mathbf{n}\) is normal vector
- \(d\) is signed distance from origin, scaled by length of \(\mathbf{n}\)

- Planes are also homogeneous
  - Any scalar multiple is same plane
Implicit Planes

\[ f = (n_x, n_y, n_z, d) \]

\[ d = -\mathbf{n} \cdot \mathbf{p} \]
Plücker Coordinates

- Parametric form of line:
  \[ L(t) = p + tv \]

- Plücker coordinates give implicit line:
  \[ v = q - p \]
  \[ m = p \times q \]
Plücker Coordinates

- $\mathbf{v}$ is the direction of the line
- $\mathbf{m}$ is the moment of the line
- Always true that $\mathbf{v} \cdot \mathbf{m} = 0$
- Representation contains no information about the points used to create the line
Direction and Moment
Plücker Coordinates

• Lots of formulas
• But little explanation

• Point \((p \mid w)\)
• Plane \([n \mid d]\)
• Line \(\{v \mid m\}\)

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A ({v \mid p \times v})</td>
<td>Line through point (p) with direction (v).</td>
</tr>
<tr>
<td>B ({p_2 - p_1 \mid p_1 \times p_2})</td>
<td>Line through two points (p_1) and (p_2).</td>
</tr>
<tr>
<td>C ({p \mid 0})</td>
<td>Line through point (p) and origin.</td>
</tr>
<tr>
<td>D ({w_2 p_2 - w_1 p_1 \mid p_1 \times p_2})</td>
<td>Line through two homogeneous points ((p_1 \mid w_1)) and ((p_2 \mid w_2)).</td>
</tr>
<tr>
<td>E ({n_2 \times n_1 \mid d_2 - d_1, n_1})</td>
<td>Line where two planes ([n \mid d]) and ([n_1 \mid d_1]) intersect.</td>
</tr>
<tr>
<td>F ({m \times n + d \mid -n \times v})</td>
<td>Homogeneous point where line ({v \mid m}) intersects plane ([n \mid d]).</td>
</tr>
<tr>
<td>G ({v \times m \mid v^2})</td>
<td>Homogeneous point closest to origin on line ({v \mid m}).</td>
</tr>
<tr>
<td>H ({-dn \mid n^2})</td>
<td>Homogeneous point closest to origin on plane ([n \mid d]).</td>
</tr>
<tr>
<td>I ({v \times u - u \times m})</td>
<td>Plane containing line ({v \mid m}) and parallel to direction (u).</td>
</tr>
<tr>
<td>J ({v \times p + m \mid -p \times m})</td>
<td>Plane containing line ({v \mid m}) and point (p).</td>
</tr>
<tr>
<td>K ([m \mid 0])</td>
<td>Plane containing line ({v \mid m}) and origin.</td>
</tr>
<tr>
<td>L ({v \times p + w m \mid -p \times m})</td>
<td>Plane containing line ({v \mid m}) and homogeneous point ((p \mid w)).</td>
</tr>
<tr>
<td>M ([m \times v \mid m^2])</td>
<td>Plane farthest from origin containing line ({v \mid m}).</td>
</tr>
<tr>
<td>N ([-wp \mid p^2])</td>
<td>Plane farthest from origin containing point ((p \mid w)).</td>
</tr>
<tr>
<td>O (\frac{{v \times m_1 + v \times m_2}}{|v \times v_2|})</td>
<td>Distance between two lines ({v \mid m_1}) and ({v_2 \mid m_2}).</td>
</tr>
<tr>
<td>P (\frac{|v \times p + m|}{|v|})</td>
<td>Distance from line ({v \mid m}) to point (p).</td>
</tr>
<tr>
<td>Q (\frac{|m|}{|v|})</td>
<td>Distance from line ({v \mid m}) to origin.</td>
</tr>
<tr>
<td>R (\frac{|p + d|}{|n|})</td>
<td>Distance from plane ([n \mid d]) to point (p).</td>
</tr>
<tr>
<td>S (\frac{|d|}{|n|})</td>
<td>Distance from plane ([n \mid d]) to origin.</td>
</tr>
</tbody>
</table>

Table from Foundations of Game Engine Development, Volume 1: Mathematics, Section 3.5.2.
Quaternions

- Encodes arbitrary rotation about origin:

\[ q = \cos \phi + a \sin \phi \]

- This is a rotation through the angle $2\phi$ about the unit-length axis $a$. 
Quaternions

- Quaternions often written as

\[ q = w + xi + yj + zk \]

- Conjugate negates “imaginary” parts:

\[ \tilde{q} = w - xi - yj - zk \]
Quaternions

- Units $i$, $j$, and $k$ multiply as follows:

  \[ i^2 = -1 \quad \quad ij = k \]
  \[ j^2 = -1 \quad \quad jk = i \]
  \[ k^2 = -1 \quad \quad ki = j \]
Quaternions

• A vector $\mathbf{v}$ is rotated by the sandwich product:

$$
\mathbf{v}' = q\mathbf{v}\tilde{q}
$$

• where $\mathbf{v}$ is regarded as the quaternion

$$
\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}
$$
Dual Quaternions

- Quaternions can rotate only about the origin
- They cannot handle translations
- Just like a $3 \times 3$ matrix

- Dual quaternions incorporate translations
- This also allows rotation about arbitrary lines
- Analogous to $4 \times 4$ matrices
Dual Quaternions

- Dual quaternion conventionally written as a pair of quaternions:

\[ \mathbf{q}_r + \epsilon \mathbf{q}_d \]

- \( \mathbf{q}_r \) is the real part
- \( \mathbf{q}_d \) is the dual part
- \( \epsilon \) squares to zero: \( \epsilon^2 = 0 \)
Dual Quaternions

- A point $\mathbf{p}$ is transformed by a dual quaternion by first writing $\mathbf{p}$ as

$$\mathbf{p} = 1 + \varepsilon\mathbf{i}p_x + \varepsilon\mathbf{j}p_y + \varepsilon\mathbf{k}p_z$$

- Then, the sandwich product is applied:

$$\mathbf{p}' = (\mathbf{q}_r + \varepsilon\mathbf{q}_d) (1 + \varepsilon\mathbf{i}p_x + \varepsilon\mathbf{j}p_y + \varepsilon\mathbf{k}p_z) (\mathbf{q}_r + \varepsilon\mathbf{q}_d)$$
Hacks!

- The dual quaternion transformation technique is an ugly hack
  - We will see that points are being cast to translation operators, transformed, and then cast back to points

- Quaternion rotations are a lesser hack, but still a hack
  - Vectors are being cast to bivectors
Hacks!

- Conventional dual quaternion methods do not handle other types of objects
  - Like lines and planes

- We are going to fix this and fill in some giant holes in the theory
What About Reflections?

- Dual quaternions give us rotations and translations
- The full set of Euclidean isometries includes improper transformations
  - Reflections
  - Inversions
  - Transflections
  - Rotoreflections
Proper Euclidean Isometries

Translation

Rotation

General screw motion
Improper Euclidean Isometries

- Reflection
- Inversion
- Transflection
- General roto-reflection
Projective Geometric Algebra (PGA)

- A four-dimensional projective space

- Point, line, and plane representations
  - With operations for combining in various ways

- Natural operations for all Euclidean isometries
  - Works with everything in the algebra
  - Both proper and improper transformations
Grassmann Algebra

- Also called exterior algebra
- Contains everything in the geometric algebra

- Fundamental geometric operations
  - Combine geometries with join and meet operations
  - Perform projection of one geometry onto another

- Isometries are part of full geometric algebra
**Wedge Product**

- Also known as exterior product
  - Grassmann called it progressive combinatorial product

- Written with upward wedge:
  \[ a \wedge b \]

- Read as “a wedge b”
Wedge Product

- The square of a vector is always zero: $\mathbf{v} \wedge \mathbf{v} = 0$

- This implies that vectors anticommute:

\[
\begin{align*}
(a + b) \wedge (a + b) &= 0 \\
\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} &= 0 \\
\mathbf{a} \wedge \mathbf{b} &= -\mathbf{b} \wedge \mathbf{a}
\end{align*}
\]
Bivectors

- Wedge product of two vectors is a “bivector”
  - Distinct from scalar or vector
  - Represents an oriented 2D area
    - Whereas a vector represents an oriented 1D direction
Bivectors

- A bivector is two directions and a magnitude
Trivectors

- Wedge product of three vectors is a “trivector”
  - Another distinct type of object
  - Represents an oriented 3D volume
  - Three directions and a magnitude
Trivectors

\[ a \wedge b \wedge c \]

\[ b \wedge c \wedge a \]

\[ c \wedge a \wedge b \]

\[ b \wedge a \wedge c \]

\[ c \wedge b \wedge a \]

\[ a \wedge c \wedge b \]
Basis Elements in 4D Space

- Four basis vectors: $e_1, e_2, e_3, e_4$
- Six basis bivectors: $e_{41}, e_{42}, e_{43}, e_{23}, e_{31}, e_{12}$
- Four basis trivectors: $e_{234}, e_{314}, e_{124}, e_{321}$
Antivectors

- Vectors use basis elements having one dimension each:

\[ x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_4 \]

- Antivectors use basis elements having all except one dimension each:

\[ x\mathbf{e}_{234} + y\mathbf{e}_{314} + z\mathbf{e}_{124} + w\mathbf{e}_{321} \]
Scalars and Antiscalars

- There are two subspaces of single-component quantities, called scalars and antiscalars
- Scalars include no dimensions of space
- Antiscalars include all dimensions of space
Scalars and Antiscalars

- We represent the scalar basis element by a bold number one: \( \mathbf{1} \)

- We represent the antiscalar basis element by a blackboard bold bold number one: \( \mathbb{1} \)

- “Anti-one”

\[ \mathbb{1} = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \]
Grade and Antigrade

- The grade of an element is the number of dimensions used by its components

- The antigrade of an element is the number of dimensions not used by its components

- These, of course, always sum to the total dimension of the algebra
## Basis Elements

<table>
<thead>
<tr>
<th>Type</th>
<th>Basis Elements</th>
<th>Grade / Antigrade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar</td>
<td>1</td>
<td>0 / 4</td>
</tr>
<tr>
<td>Vectors</td>
<td>e₁, e₂, e₃, e₄</td>
<td>1 / 3</td>
</tr>
<tr>
<td>Bivectors</td>
<td>e₂₃ = e₂ ∧ e₃, e₃₁ = e₃ ∧ e₁, e₁₂ = e₁ ∧ e₂, e₄₃ = e₄ ∧ e₃, e₄₂ = e₄ ∧ e₂, e₄₁ = e₄ ∧ e₁</td>
<td>2 / 2</td>
</tr>
<tr>
<td>Trivectors / Antivectors</td>
<td>e₃₂₁ = e₃ ∧ e₂ ∧ e₁, e₁₂₄ = e₁ ∧ e₂ ∧ e₄, e₃₁₄ = e₃ ∧ e₁ ∧ e₄, e₂₃₄ = e₂ ∧ e₃ ∧ e₄</td>
<td>3 / 1</td>
</tr>
<tr>
<td>Antiscalar</td>
<td>1 = e₁ ∧ e₂ ∧ e₃ ∧ e₄</td>
<td>4 / 0</td>
</tr>
</tbody>
</table>
Homogeneous Point

- Ordinary vector

\[ \mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4 \]
Homogeneous Point

- Projection of 1D vector into subspace at $w = 1$ is a 0D point
Point at Infinity

- If $w$ coordinate is zero, then vector represents a point at infinity in the $(x, y, z)$ direction

- Each point at infinity exists in one direction
Homogeneous Line

- Wedge product of two points is a bivector

\[
p \wedge q = (q_x p_w - p_x q_w) e_{41} + (q_y p_w - p_y q_w) e_{42} + (q_z p_w - p_z q_w) e_{43} \\
+ (p_y q_z - p_z q_y) e_{23} + (p_z q_x - p_x q_z) e_{31} + (p_x q_y - p_y q_x) e_{12}
\]

\[
L = v_x e_{41} + v_y e_{42} + v_z e_{43} + m_x e_{23} + m_y e_{31} + m_z e_{12}
\]

\[
\text{Direction} \quad \text{Moment}
\]
Homogeneous Line

- Projection of 2D bivector into subspace at $w = 1$ is a 1D line
Line at Infinity

- If direction part is zero, then line lies at infinity in directions perpendicular to moment

- Each line at infinity exists in a plane of directions
Line at Infinity

\[ L = m_x e_{23} + m_y e_{31} + m_z e_{12} \]
Homogeneous Plane

- Wedge product of three points is a trivector

\[ \mathbf{f} = f_x \mathbf{e}_{234} + f_y \mathbf{e}_{314} + f_z \mathbf{e}_{124} + f_w \mathbf{e}_{321} \]

Normal
Homogeneous Plane

- Projection of 3D trivector into subspace at $w = 1$ is a 2D plane
Plane at Infinity

- There is one plane at infinity
  - Just like there is one point at the origin
  - These are duals of each other

- The plane at infinity exists in all directions
Bulk and Weight

- Components of any object can be separated into two parts

- The “bulk” consists of all components that do not have a factor of $e_4$

- The “weight” consists of all components that do have a factor of $e_4$
Bulk and Weight

- The bulk of $a$ is denoted by $a\bullet$
- The weight of $a$ is denoted by $a\circ$
- Any object is the sum of its bulk and weight:

$$a = a\bullet + a\circ$$
Bulk and Weight

- Weight of point is its $w$ coordinate
- Weight of line is its direction
- Weight of plane is its normal

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
<th>Bulk</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>$p = p_x e_1 + p_y e_2 + p_z e_3 + p_w e_4$</td>
<td>$p_\bullet = p_x e_1 + p_y e_2 + p_z e_3$</td>
<td>$p_O = p_w e_4$</td>
</tr>
<tr>
<td>Line</td>
<td>$L = v_x e_{41} + v_y e_{42} + v_z e_{43} + m_x e_{23} + m_y e_{31} + m_z e_{12}$</td>
<td>$L_\bullet = m_x e_{23} + m_y e_{31} + m_z e_{12}$</td>
<td>$L_O = v_x e_{41} + v_y e_{42} + v_z e_{43}$</td>
</tr>
<tr>
<td>Plane</td>
<td>$f = f_x e_{234} + f_y e_{314} + f_z e_{124} + f_w e_{321}$</td>
<td>$f_\bullet = f_w e_{321}$</td>
<td>$f_O = f_x e_{234} + f_y e_{314} + f_z e_{124}$</td>
</tr>
</tbody>
</table>
Bulk and Weight

- The bulk contains an object’s position
- The weight contains attitude and orientation

- An object with zero bulk contains the point at the origin
- An object with zero weight is contained by the plane at infinity
Duality

- There is a fundamental symmetry in geometric algebra

- We have assigned dimensionality to objects based on how many basis vectors are \textit{present}

- Objects have another dimensionality based on how many basis vectors are \textit{absent}
Duality

- Every object is really two things at once
  - Full space and empty space
  - Grade and antigrade
- This is duality, and it’s everywhere in GA

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<td>$1$</td>
<td>$0 / 4$</td>
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<tr>
<td>Vectors</td>
<td>$e_1$ $e_2$ $e_3$</td>
<td>$1 / 3$</td>
</tr>
<tr>
<td>Bivectors</td>
<td>$e_{23} = e_2 \wedge e_3$ $e_{31} = e_3 \wedge e_1$ $e_{12} = e_1 \wedge e_2$ $e_{43} = e_4 \wedge e_3$ $e_{42} = e_4 \wedge e_2$ $e_{41} = e_4 \wedge e_1$</td>
<td>$2 / 2$</td>
</tr>
<tr>
<td>Trivectors / Antivectors</td>
<td>$e_{321} = e_3 \wedge e_2 \wedge e_1$ $e_{124} = e_1 \wedge e_2 \wedge e_4$ $e_{314} = e_3 \wedge e_1 \wedge e_4$ $e_{234} = e_2 \wedge e_3 \wedge e_4$</td>
<td>$3 / 1$</td>
</tr>
<tr>
<td>Antiscalar</td>
<td>$1 = e_1 \wedge e_2 \wedge e_3 \wedge e_4$</td>
<td>$4 / 0$</td>
</tr>
</tbody>
</table>
Duality

- A point has one full dimension
- It also has three empty dimensions
- From different perspectives, it simultaneously looks like a point and a plane
Duality
Dualization

- We can map basis elements so that full and empty dimensions are exchanged
- If we think of the dimensions used by a basis element as a 4-bit code, then dualization inverts the bits
- There are many choices for dualization functions, and they just differ in sign in a grade-dependent manner
Complements

● One choice for dualization is the “right complement”

● The right complement of $a$ is the object $\bar{a}$ such that

$$a \wedge \bar{a} = 1$$

● This is also called the Hodge dual
Complements

- In 4D, right complement is not an involution
- The inverse is the "left complement" \( \mathbf{a} \)
  \[ \mathbf{a} \wedge \mathbf{a} = 1 \]
- Right and left complements differ only in sign
Complements

- Here, the basis elements are ordered so that taking the complement just reverses the list and adjusts the sign.

<table>
<thead>
<tr>
<th>Basis element $\mathbf{a}$</th>
<th>1</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
<th>$e_{23}$</th>
<th>$e_{31}$</th>
<th>$e_{12}$</th>
<th>$e_{43}$</th>
<th>$e_{42}$</th>
<th>$e_{41}$</th>
<th>$e_{321}$</th>
<th>$e_{124}$</th>
<th>$e_{314}$</th>
<th>$e_{234}$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Right complement $\overline{\mathbf{a}}$</td>
<td>1</td>
<td>$e_{234}$</td>
<td>$e_{314}$</td>
<td>$e_{124}$</td>
<td>$e_{321}$</td>
<td>$-e_{41}$</td>
<td>$-e_{42}$</td>
<td>$-e_{43}$</td>
<td>$-e_{12}$</td>
<td>$-e_{31}$</td>
<td>$-e_{23}$</td>
<td>$-e_{4}$</td>
<td>$-e_{3}$</td>
<td>$-e_{2}$</td>
<td>$-e_{1}$</td>
<td>1</td>
</tr>
<tr>
<td>Left complement $\mathbf{a}$</td>
<td>1</td>
<td>$-e_{234}$</td>
<td>$-e_{314}$</td>
<td>$-e_{124}$</td>
<td>$-e_{321}$</td>
<td>$-e_{41}$</td>
<td>$-e_{42}$</td>
<td>$-e_{43}$</td>
<td>$-e_{12}$</td>
<td>$-e_{31}$</td>
<td>$-e_{23}$</td>
<td>$e_{4}$</td>
<td>$e_{3}$</td>
<td>$e_{2}$</td>
<td>$e_{1}$</td>
<td>1</td>
</tr>
<tr>
<td>Double complement $\overline{\overline{\mathbf{a}}}$ or $\overline{\mathbf{a}}$</td>
<td>1</td>
<td>$-e_1$</td>
<td>$-e_2$</td>
<td>$-e_3$</td>
<td>$-e_4$</td>
<td>$e_{23}$</td>
<td>$e_{31}$</td>
<td>$e_{12}$</td>
<td>$e_{43}$</td>
<td>$e_{42}$</td>
<td>$e_{41}$</td>
<td>$-e_{321}$</td>
<td>$-e_{124}$</td>
<td>$-e_{314}$</td>
<td>$-e_{234}$</td>
<td>1</td>
</tr>
</tbody>
</table>
Attitude Extraction

- Weight contains information about attitude
- The weight complement is useful for extracting this information to be used another way
- Very useful for projections

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<th>Weight Complement</th>
<th>Interpretation</th>
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<tbody>
<tr>
<td>Point</td>
<td>$p = p_x e_1 + p_y e_2 + p_z e_3 + p_w e_4$</td>
<td>$p_O = -p_w e_{321}$</td>
<td>Plane at infinity.</td>
</tr>
<tr>
<td>Line</td>
<td>$L = v_x e_{41} + v_y e_{42} + v_z e_{43} + m_x e_{31} + m_y e_{31} + m_z e_{12}$</td>
<td>$L_O = -v_x e_{31} - v_y e_{31} - v_z e_{12}$</td>
<td>Line at infinity perpendicular to line $L$.</td>
</tr>
<tr>
<td>Plane</td>
<td>$f = f_x e_{234} + f_y e_{314} + f_z e_{124} + f_w e_{321}$</td>
<td>$f_O = f_x e_1 + f_y e_2 + f_z e_3$</td>
<td>Normal vector of the plane $f$.</td>
</tr>
</tbody>
</table>
Antiwedge Product

- Also known as exterior antiproduct
  - Grassmann called it regressive combinatorial product

- Written with downward wedge:
  \[ a \vee b \]

- Read as “a antiwedge b”
Antiwedge Product

- **Wedge product combines full dimensions**
  - Add grades of operands

- **Antiwedge product combines empty dimensions**
  - Add antigrades of operands
Antiwedge Product

- Dual to wedge product
  \[ \mathbf{c} = \mathbf{a} \land \mathbf{b} \]
  \[ \overline{\mathbf{c}} = \overline{\mathbf{a}} \lor \overline{\mathbf{b}} \]

- Operates on antivectors in the same way that wedge product operates on vectors.
De Morgan’s Laws

- All operations in GA have duals that together satisfy De Morgan’s Laws
- For wedge and antiwedge:
  \[ a \lor b = \overline{a} \land \overline{b} = a \land b \]
- This can be taken as definition of antiwedge
  - Depends on specific choice of dualization function
  - Only affects orientation of some results
Join and Meet

- **Wedge product combines full dimensions**
  - Join operation
  - Analogous to union

- **Antiwedge product combines empty dimensions**
  - Meet operation
  - Analogous to intersection
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<tbody>
<tr>
<td>$p \wedge q = (q, p_x - p_x, v, q_x) e_4 + (q, p_x - p_x, p_y, q_y, e_4 + (q, p_x - p_x, q_z, e_4) + (p_x, p_y - p_y, v, p_z - p_z)) e_z$</td>
<td>Line containing points $p$ and $q$. Zero if $p$ and $q$ are coincident.</td>
<td>![p ∧ q]</td>
</tr>
<tr>
<td>$L \wedge p = (v, p_x - v, p_y + m, p_z) e_{24} + (v, p_x - v, p_y - m, p_z) e_{24}$</td>
<td>Plane containing line $L$ and point $p$. Normal is zero if $p$ lies in $L$.</td>
<td>![L ∧ p]</td>
</tr>
<tr>
<td>$f \lor g = (f, g_x - f, g_y) e_4 + (f, g_x - f, g_y) e_4 + (f, g_x - f, g_y) e_4 + (f, g_x - f, g_y) e_4$</td>
<td>Plane where planes $f$ and $g$ intersect. Direction is zero if $f$ and $g$ are parallel.</td>
<td>![f ∨ g]</td>
</tr>
<tr>
<td>$L \lor f = (m, f_x - m, f_y + v, f_z) e_4 + (m, f_x - m, f_y + v, f_z) e_4 + (m, f_x - m, f_y + v, f_z) e_4$</td>
<td>Point where line $L$ intersects plane $f$. Weight is zero if $L$ and $f$ are parallel.</td>
<td>![L ∨ f]</td>
</tr>
<tr>
<td>$f_\perp p = -f_x p_x e_4 - f_y p_y e_4 - f_z p_z e_4 + (f, p_x - f, p_x) e_4 + (f, p_y - f, p_y) e_4 + (f, p_z - f, p_z) e_4$</td>
<td>Line perpendicular to plane $f$ and passing through point $p$.</td>
<td>![f_⊥ p]</td>
</tr>
<tr>
<td>$L_\perp p = -v_x p_x e_{24} - v_y p_y e_{24} - v_z p_z e_{24}$</td>
<td>Plane perpendicular to line $L$ and containing point $p$.</td>
<td>![L_⊥ p]</td>
</tr>
<tr>
<td>$f_\perp L = (v, f_x - v, f_y) e_{24} + (v, f_x - v, f_y) e_{24} + (v, f_x - v, f_y) e_{24}$</td>
<td>Plane perpendicular to plane $f$ and containing line $L$. Normal is zero if $L$ is perpendicular to $f$.</td>
<td>![f_⊥ L]</td>
</tr>
</tbody>
</table>
Plane/Point Volume

- Wedge product of point $\mathbf{p}$ and plane $\mathbf{f}$ is
  
  $$\mathbf{p} \wedge \mathbf{f} = (p_x f_x + p_y f_y + p_z f_z + p_w f_w) \mathbf{1}$$

- Same as conventional dot product

- Gives signed distance between point and plane, scaled by weights of point and plane
Line/Line Volume

- Wedge product of two lines $L_1$ and $L_2$ is
  \[ L_1 \wedge L_2 = - (v_1 \cdot m_2 + v_2 \cdot m_1) \mathbf{1} \]

- Gives signed distance between lines, scaled by magnitude of $\mathbf{v}_1 \wedge \mathbf{v}_2$
Line Crossing

- Antiwedge product gives same value as scalar
- Used to detect which way lines cross each other

\[
\mathbf{L}_1 \vee \mathbf{L}_2 > 0 \quad \text{and} \quad \mathbf{L}_1 \vee \mathbf{L}_2 < 0
\]
Line Between Two Lines

- Line $J$ perpendicular to lines $K$ and $L$
  - Can’t be produced by wedge/antiwedge product
  - It does appear in the geometric product
Application: Shadow Regions

- Need convex region where shadow castors must be to affect scene
- Precompute lines for frustum edges
- Find silhouette w.r.t. light
- Take wedge products with light position
Projections

- Wedge and antiwedge products in specific combinations perform projections

- These are derived from “interior products”
  - All projections have a uniform formula
  - Interior antiproducts perform “antiprojections”
# Projections

<table>
<thead>
<tr>
<th>Projection</th>
<th>Illustration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Projection of point ( p ) onto plane ( f ).</td>
<td><img src="image1.png" alt="Diagram" /></td>
</tr>
<tr>
<td>((f \odot \wedge p) \vee f = (f_x^2 + f_y^2 + f_z^2) p - (f_x p_x + f_y p_y + f_z p_z + f_w p_w) (f_x e_1 + f_y e_2 + f_z e_3))</td>
<td><img src="image2.png" alt="Diagram" /></td>
</tr>
<tr>
<td>Projection of point ( p ) onto line ( L ).</td>
<td><img src="image3.png" alt="Diagram" /></td>
</tr>
<tr>
<td>((L \odot \wedge p) \vee L = (v_x p_x + v_y p_y + v_z p_z) v + (v_x^2 + v_y^2 + v_z^2) p_w e_4 + (v_x m_z - v_z m_y) p_u e_1 + (v_y m_z - v_z m_x) p_u e_2 + (v_x m_y - v_y m_z) p_u e_3)</td>
<td><img src="image4.png" alt="Diagram" /></td>
</tr>
<tr>
<td>Projection of line ( L ) onto plane ( f ).</td>
<td><img src="image5.png" alt="Diagram" /></td>
</tr>
<tr>
<td>((f \odot \wedge L) \vee f = (f_x^2 + f_y^2 + f_z^2)(v_x e_{41} + v_y e_{42} + v_z e_{43}) - (f_x v_x + f_y v_y + f_z v_z)(f_x e_{41} + f_y e_{42} + f_z e_{43}) + (f_x m_x + f_y m_y + f_z m_z)(f_x e_{23} + f_y e_{31} + f_z e_{12}) - (f_y v_z - f_z v_y) f_w e_{23} - (f_z v_x - f_x v_z) f_w e_{31} - (f_x v_y - f_y v_x) f_w e_{12})</td>
<td><img src="image6.png" alt="Diagram" /></td>
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## Antiprojections

<table>
<thead>
<tr>
<th>Antiprojection</th>
<th>Illustration</th>
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<tbody>
<tr>
<td>Antiprojection of plane ( f ) onto point ( p ).</td>
<td><img src="image1" alt="Illustration" /></td>
</tr>
<tr>
<td>((p \odot \vee f) \wedge p = f_x p_w^2 e_{234} + f_y p_w^2 e_{314} + f_z p_w^2 e_{124} - (f_x p_x + f_y p_y + f_z p_z) p_w e_{321})</td>
<td></td>
</tr>
<tr>
<td>Antiprojection of line ( L ) onto point ( p ).</td>
<td><img src="image2" alt="Illustration" /></td>
</tr>
<tr>
<td>((p \odot \vee L) \wedge p = v_x p_w^2 e_{41} + v_y p_w^2 e_{42} + v_z p_w^2 e_{43} + (p_y v_z - p_z v_y) p_w e_{23} + (p_z v_x - p_x v_z) p_w e_{31} + (p_x v_y - p_y v_x) p_w e_{12})</td>
<td></td>
</tr>
<tr>
<td>Antiprojection of plane ( f ) onto line ( L ).</td>
<td><img src="image3" alt="Illustration" /></td>
</tr>
<tr>
<td>((L \odot \vee f) \wedge L = (v_x^2 + v_y^2 + v_z^2)(f_x e_{234} + f_y e_{314} + f_z e_{124}) - (f_x v_x + f_y v_y + f_z v_z)(v_x e_{234} + v_y e_{314} + v_z e_{124}) + (f_x m_y v_z - f_y m_z v_y + f_y m_z v_x - f_x m_x v_y + f_z m_x v_y - f_z m_y v_x) e_{321})</td>
<td></td>
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</table>
## Special Projections

- **Point at origin and plane at infinity produce special values**

<table>
<thead>
<tr>
<th>Projection</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>[(\mathbf{f}_o \land \mathbf{e}_4) \lor \mathbf{f} = -f_x f_w e_1 - f_y f_w e_2 - f_z f_w e_3 + (f_x^2 + f_y^2 + f_z^2) e_4]</td>
<td>Point closest to the origin on the plane (\mathbf{f}).</td>
</tr>
<tr>
<td>[(\mathbf{L}_o \land \mathbf{e}_4) \lor \mathbf{L} = (v_x m_z - v_z m_x) e_1 + (v_y m_z - v_z m_y) e_2 + (v_z m_y - v_y m_z) e_3 + (v_x^2 + v_y^2 + v_z^2) e_4]</td>
<td>Point closest to the origin on the line (\mathbf{L}).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Antiprojection</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>[(\mathbf{p} \lor \mathbf{e}<em>{321}) \land \mathbf{p} = -p_x p_w e</em>{234} - p_y p_w e_{314} - p_z p_w e_{124} + (p_x^2 + p_y^2 + p_z^2) e_{321}]</td>
<td>Plane farthest from the origin containing the point (\mathbf{p}).</td>
</tr>
<tr>
<td>[(\mathbf{L} \lor \mathbf{e}<em>{321}) \land \mathbf{L} = (m_x v_z - m_z v_x) e</em>{234} + (m_y v_z - m_z v_y) e_{314} + (m_z v_y - m_y v_z) e_{124} + (m_x^2 + m_y^2 + m_z^2) e_{321}]</td>
<td>Plane farthest from the origin containing the line (\mathbf{L}).</td>
</tr>
</tbody>
</table>
Geometric Product

- Adds more information to wedge product
- Incorporates a metric
  - Allows us to make measurements
  - Provides the mechanism for Euclidean isometries

- Like all operations in GA, the geometric product has a dual operation, or antiproduct
Geometric Product

- Conventional treatments of GA ignore the antiproduct
- Geometric product has been expressed by plain old juxtaposition: $c = ab$
- With two products, we need an infix symbol to distinguish between them
Geometric Product

- The geometric product incorporates the wedge product and adds information to it.
- So we write the geometric product as:

\[ a \wedge b \]

- We read this as “a wedge-dot b”
Geometric Antiproduct

- The geometric antiproduct incorporates the antiwedge product and adds information to it.
- So we write the geometric antiproduct as $a \vee b$.
- We read this as “a antiwedge-dot b”.
The 4D projective geometric algebra is denoted by $\mathbb{G}_{3,0,1}$

The subscripts mean that:
- 3 basis vectors square to $+1$
- 0 basis vectors square to $-1$
- 1 basis vector squares to 0

The fourth dimension has no physical measure
Metric

- Metrics apply symmetrically to geometric product and antiproduct

\[
\begin{align*}
\mathbf{e}_1 \wedge \mathbf{e}_1 &= 1 \\
\mathbf{e}_2 \wedge \mathbf{e}_2 &= 1 \\
\mathbf{e}_3 \wedge \mathbf{e}_3 &= 1 \\
\mathbf{e}_4 \wedge \mathbf{e}_4 &= 0 \\
\bar{\mathbf{e}}_1 \vee \bar{\mathbf{e}}_1 &= 1 \\
\bar{\mathbf{e}}_2 \vee \bar{\mathbf{e}}_2 &= 1 \\
\bar{\mathbf{e}}_3 \vee \bar{\mathbf{e}}_3 &= 1 \\
\bar{\mathbf{e}}_4 \vee \bar{\mathbf{e}}_4 &= 0
\end{align*}
\]
## Geometric Product

**Geometric Product** \( a \wedge b \)

\[
\begin{array}{cccccccccccccc}
 a & b & 1 & e_1 & e_2 & e_3 & e_4 & e_{23} & e_{31} & e_{12} & e_{43} & e_{42} & e_{41} & e_{321} & e_{124} & e_{314} & e_{234} & 1 \\
 1 & 1 & e_1 & e_2 & e_3 & e_4 & e_{23} & e_{31} & e_{12} & e_{43} & e_{42} & e_{41} & e_{321} & e_{124} & e_{314} & e_{234} & 1 \\
e_1 & e_1 & 1 & e_{12} & -e_{31} & -e_{41} & -e_{321} & -e_{3} & e_2 & e_{314} & -e_{124} & -e_4 & -e_{23} & -e_{42} & e_{43} & 1 & e_{234} \\
e_2 & e_2 & -e_{12} & 1 & e_{33} & -e_{42} & e_3 & -e_{321} & -e_1 & -e_{234} & -e_4 & e_{124} & -e_{31} & e_{41} & 1 & -e_{43} & e_{314} \\
e_3 & e_3 & e_{31} & -e_{23} & 1 & -e_{43} & -e_2 & e_1 & -e_{321} & -e_4 & e_{234} & -e_{314} & -e_{12} & 1 & -e_{41} & e_{42} & e_{124} \\
e_4 & e_4 & e_{41} & e_{42} & e_{43} & 0 & e_{234} & e_{314} & e_{124} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
e_{23} & e_{23} & -e_{321} & -e_1 & e_2 & e_{234} & -1 & -e_{12} & e_{31} & e_{42} & -e_{43} & -1 & 1 & e_1 & e_{314} & -e_{124} & -e_4 & e_{41} \\
e_{31} & e_{31} & e_3 & -e_{321} & -e_1 & e_{314} & e_{12} & -1 & -e_{23} & -e_{41} & -1 & 1 & e_3 & e_{2} & -e_{234} & -e_4 & e_{124} & e_{42} \\
e_{12} & e_{12} & -e_2 & e_1 & -e_{231} & e_{124} & -e_{31} & e_{23} & -1 & -1 & e_{43} & -e_{42} & e_3 & -e_4 & e_{234} & -e_{314} & e_{43} \\
e_{43} & e_{43} & e_{314} & -e_{234} & e_6 & 0 & -e_{42} & e_{41} & -1 & 0 & 0 & 0 & -e_{124} & 0 & 0 & 0 & 0 \\
e_{42} & e_{42} & -e_{124} & e_4 & e_{234} & 0 & e_{43} & -1 & -e_{41} & 0 & 0 & 0 & -e_{314} & 0 & 0 & 0 & 0 \\
e_{41} & e_{41} & e_4 & e_{124} & -e_{234} & 0 & -1 & -e_{43} & e_{42} & 0 & 0 & 0 & -e_{234} & 0 & 0 & 0 & 0 \\
e_{321} & e_{321} & -e_{23} & -e_{31} & -e_{12} & -1 & e_1 & e_2 & e_3 & e_{124} & e_{314} & e_{234} & -1 & -e_{43} & -e_{42} & -e_{31} & e_4 \\
e_{124} & e_{124} & -e_{42} & e_{41} & -1 & 0 & -e_{314} & e_{234} & -e_4 & 0 & 0 & 0 & e_{43} & 0 & 0 & 0 & 0 \\
e_{314} & e_{314} & e_{43} & -1 & -e_{41} & 0 & e_{124} & -e_4 & -e_{234} & 0 & 0 & 0 & e_{42} & 0 & 0 & 0 & 0 \\
e_{234} & e_{234} & -1 & -e_{43} & e_{42} & 0 & -e_4 & -e_{124} & e_{314} & 0 & 0 & 0 & e_{41} & 0 & 0 & 0 & 0 \\
1 & 1 & -e_{234} & -e_{314} & -e_{124} & 0 & e_{41} & e_{42} & e_{43} & 0 & 0 & 0 & -e_4 & 0 & 0 & 0 & 0 \\
\end{array}
\]
# Geometric Antiproduct

## Geometric Antiproduct $a \bigtriangledown b$

<table>
<thead>
<tr>
<th>$a \bigtriangledown b$</th>
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<td>$e_{124}$</td>
<td>$-e_{4}$</td>
<td>$e_{324}$</td>
<td>$-e_{4}$</td>
<td>$-e_{124}$</td>
<td>$e_{31}$</td>
<td>$-e_{41}$</td>
</tr>
</tbody>
</table>

1 | 1 | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_{23}$ | $e_{31}$ | $e_{32}$ | $e_{43}$ | $e_{42}$ | $e_{321}$ | $e_{124}$ | $e_{314}$ | $e_{234}$ | 1
Geometric Product and Antiproduct
Geometric Product and Antiproduct

- $1$ is the multiplicative identity of the product
  
  $$1 \wedge a = a \wedge 1 = a$$

- $1$ is the multiplicative identity of the antiproduct
  
  $$1 \vee a = a \vee 1 = a$$
Reverse

- Unary operation called “reverse” rearranges vector basis element factors so they’re multiplied in reverse order

- If this results in an odd permutation, then the effect is that the term is negated

- Mechanism underlying conjugate operation
Antireverse

- As with everything in GA, the reverse has a dual operation, the “antireverse”

- The antireverse rearranges factors so that antivector basis elements are multiplied in reverse order under the antiproduct
Reverses

- The reverse of $a$ is written $\hat{a}$
- The antireverse of $a$ is written $\tilde{a}$

<table>
<thead>
<tr>
<th>Basis element $a$</th>
<th>1</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
<th>$e_{23}$</th>
<th>$e_{31}$</th>
<th>$e_{12}$</th>
<th>$e_{43}$</th>
<th>$e_{42}$</th>
<th>$e_{321}$</th>
<th>$e_{124}$</th>
<th>$e_{314}$</th>
<th>$e_{234}$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reverse $\hat{a}$</td>
<td>1</td>
<td>$e_1$</td>
<td>$e_2$</td>
<td>$e_3$</td>
<td>$e_4$</td>
<td>$-e_{23}$</td>
<td>$-e_{31}$</td>
<td>$-e_{12}$</td>
<td>$-e_{43}$</td>
<td>$-e_{42}$</td>
<td>$-e_{321}$</td>
<td>$-e_{124}$</td>
<td>$-e_{314}$</td>
<td>$-e_{234}$</td>
<td>$1$</td>
</tr>
<tr>
<td>Antireverse $\tilde{a}$</td>
<td>1</td>
<td>$-e_1$</td>
<td>$-e_2$</td>
<td>$-e_3$</td>
<td>$-e_4$</td>
<td>$-e_{23}$</td>
<td>$-e_{31}$</td>
<td>$-e_{12}$</td>
<td>$-e_{43}$</td>
<td>$-e_{42}$</td>
<td>$e_{321}$</td>
<td>$e_{124}$</td>
<td>$e_{314}$</td>
<td>$e_{234}$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Reverses and Complements

- In GA, multiplication by 1 has long been used to algebraically calculate a dual.

- This doesn’t work in PGA because \( e_4 \wedge e_4 = 0 \).

- With only one product, only part of the dual gets calculated:

\[
\begin{align*}
\mathbf{a} \cdot &= \tilde{\mathbf{a}} \wedge \mathbf{1} \\
\mathbf{a} \wedge &= \tilde{\mathbf{a}} \wedge \mathbf{1}
\end{align*}
\]
Reverses and Complements

- The antiproduct is necessary for the remaining pieces of the dual:

\[ a \text{ } \bigcirc \text{ } = 1 \lor \hat{a} \]

\[ a \text{ } \bigcirc \text{ } = 1 \lor a \]

- Complete duals can now be written as

\[ \overline{a} = \hat{a} \land 1 + 1 \lor \hat{a} \]

\[ a = a \land 1 + 1 \lor a \]
Reflection Through Plane

- All isometries can be broken down into reflections through one or more planes.

- Isometries fall into two classes:
  - Even number of reflections: proper isometry
  - Odd number of reflections: improper isometry
Fundamental Operation

- Remember, all objects are two things at once

- Reflect dual point of $f$ through dual plane of $p$: $-p \wedge f \wedge \tilde{p}$

- Reflect point $p$ through plane $f$: $-f \vee p \vee f$
Fundamental Operation

- We can choose to identify objects by the dimensions that are absent/empty and use the geometric product.

- Or we can choose to identify objects by the dimensions that are present/full and use the geometric antiproduct.
Fundamental Operation

- Both methods are equally valid and produce the same results

- We choose the second option so that points, lines, and planes remain 1, 2, and 3 dimensional in projective space, respectively
Reflection Through Two Planes

- Reflection through two planes meeting at an angle $\phi$
- Rotates about line of intersection by $2\phi$
Reflection Through Two Planes

- If planes are parallel, result is a translation
Motors

- Operator that performs a general proper Euclidean isometry
  - Any combination of rotations and translations
  - Product of an even number of reflections

- Portmanteau of “motion operator” or “moment vector”
Motors

- General form:

\[ Q = r_x e_{41} + r_y e_{42} + r_z e_{43} + r_w \mathbb{1} + u_x e_{23} + u_y e_{31} + u_z e_{12} + u_w \]

Rotation

- Transformation: \( a' = Q \backslash a \backslash Q \)
Flectors

- Operator that performs a general improper Euclidean isometry
  - Any combination of a reflection with other rotations and translations
  - Product of an odd number of reflections

- Portmanteau of “reflection operator”
Flectors

- General form:

\[ \mathbf{G} = s_x \mathbf{e}_1 + s_y \mathbf{e}_2 + s_z \mathbf{e}_3 + s_w \mathbf{e}_4 + h_x \mathbf{e}_{234} + h_y \mathbf{e}_{314} + h_z \mathbf{e}_{124} + h_w \mathbf{e}_{321} \]

Point \quad Plane

- Transformation: \( \mathbf{a}' = -\mathbf{G} \lor \mathbf{a} \lor \bar{\mathbf{G}} \)
Five Types of Objects in $\mathcal{G}_{3,0,1}$

- **Point**
  - Four vector components

- **Line**
  - Six bivector components

- **Plane**
  - Four trivector components
Five Types of Objects in $\mathcal{G}_{3,0,1}$

- **Motor**
  - Eight components: scalar, bivector, antiscalar

- **Flector**
  - Eight components: vector, trivector
Motors and flectors also have bulk and weight

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
<th>Bulk</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>$\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4$</td>
<td>$\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3$</td>
<td>$\mathbf{p} = p_w \mathbf{e}_4$</td>
</tr>
<tr>
<td>Line</td>
<td>$\mathbf{L} = v_x \mathbf{e}<em>{41} + v_y \mathbf{e}</em>{42} + v_z \mathbf{e}<em>{43} + m_x \mathbf{e}</em>{23} + m_y \mathbf{e}<em>{31} + m_z \mathbf{e}</em>{12}$</td>
<td>$\mathbf{L} = m_x \mathbf{e}<em>{23} + m_y \mathbf{e}</em>{31} + m_z \mathbf{e}_{12}$</td>
<td>$\mathbf{L} = v_x \mathbf{e}<em>{41} + v_y \mathbf{e}</em>{42} + v_z \mathbf{e}_{43}$</td>
</tr>
<tr>
<td>Plane</td>
<td>$\mathbf{f} = f_x \mathbf{e}<em>{234} + f_y \mathbf{e}</em>{314} + f_z \mathbf{e}<em>{124} + f_w \mathbf{e}</em>{321}$</td>
<td>$\mathbf{f} = f_w \mathbf{e}_{321}$</td>
<td>$\mathbf{f} = f_x \mathbf{e}<em>{234} + f_y \mathbf{e}</em>{314} + f_z \mathbf{e}_{124}$</td>
</tr>
<tr>
<td>Motor</td>
<td>$\mathbf{Q} = r_x \mathbf{e}<em>{41} + r_y \mathbf{e}</em>{42} + r_z \mathbf{e}<em>{43} + r_w \mathbf{1} + u_x \mathbf{e}</em>{23} + u_y \mathbf{e}<em>{31} + u_z \mathbf{e}</em>{12} + u_w$</td>
<td>$\mathbf{Q} = u_x \mathbf{e}<em>{23} + u_y \mathbf{e}</em>{31} + u_z \mathbf{e}_{12} + u_w$</td>
<td>$\mathbf{Q} = r_x \mathbf{e}<em>{41} + r_y \mathbf{e}</em>{42} + r_z \mathbf{e}_{43} + r_w \mathbf{1}$</td>
</tr>
<tr>
<td>Flector</td>
<td>$\mathbf{G} = s_x \mathbf{e}<em>1 + s_y \mathbf{e}<em>2 + s_z \mathbf{e}<em>3 + s_w \mathbf{e}<em>4 + h_x \mathbf{e}</em>{234} + h_y \mathbf{e}</em>{314} + h_z \mathbf{e}</em>{124} + h_w \mathbf{e}</em>{321}$</td>
<td>$\mathbf{G} = s_x \mathbf{e}_1 + s_y \mathbf{e}_2 + s_z \mathbf{e}<em>3 + h_w \mathbf{e}</em>{321}$</td>
<td>$\mathbf{G} = s_w \mathbf{e}<em>4 + h_x \mathbf{e}</em>{234} + h_y \mathbf{e}<em>{314} + h_z \mathbf{e}</em>{124}$</td>
</tr>
</tbody>
</table>
Geometric Property

• Not every possible multivector $a$ is a valid geometric object

• Must satisfy $a \wedge \bar{a} = \text{scalar}$

• Equivalently $a \vee \bar{a} = \text{antiscalar}$

• All vectors and antivectors are valid
This imposes the following requirements

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
<th>Requirement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>$p = p_x e_1 + p_y e_2 + p_z e_3 + p_w e_4$</td>
<td>—</td>
</tr>
<tr>
<td>Line</td>
<td>$L = v_z e_{41} + v_y e_{42} + v_x e_{43} + m_x e_{31} + m_y e_{32} + m_z e_{12}$</td>
<td>$v_x m_x + v_y m_y + v_z m_z = 0$</td>
</tr>
<tr>
<td>Plane</td>
<td>$f = f_x e_{234} + f_y e_{314} + f_z e_{124} + f_w e_{321}$</td>
<td>—</td>
</tr>
<tr>
<td>Motor</td>
<td>$Q = r_z e_{41} + r_y e_{42} + r_x e_{43} + r_w e_{31} + u_x e_{23} + u_y e_{31} + u_z e_{12} + u_w$</td>
<td>$r_x u_x + r_y u_y + r_z u_z + r_w u_w = 0$</td>
</tr>
<tr>
<td>Flector</td>
<td>$G = s_x e_1 + s_y e_2 + s_z e_3 + s_w e_4 + h_x e_{234} + h_y e_{314} + h_z e_{124} + h_w e_{321}$</td>
<td>$s_x h_x + s_y h_y + s_z h_z + s_w h_w = 0$</td>
</tr>
</tbody>
</table>
Norms

- Since there are two geometric products, there are two different norms

- Bulk norm: \[ \| \mathbf{a} \|_\bullet = \sqrt{\mathbf{a} \wedge \tilde{\mathbf{a}}} \]

- Weight norm: \[ \| \mathbf{a} \|_\circ = \sqrt{\mathbf{a} \vee \tilde{\mathbf{a}}} \]
Norms

- Bulk norm is a scalar
- Weight norm is an antiscalar

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
<th>Bulk Norm</th>
<th>Weight Norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>( \mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4 )</td>
<td>( | \mathbf{p} |_\bullet = \sqrt{p_x^2 + p_y^2 + p_z^2} )</td>
<td>( | \mathbf{p} |_\circ =</td>
</tr>
<tr>
<td>Line</td>
<td>( \mathbf{L} = v_x \mathbf{e}<em>{41} + v_y \mathbf{e}</em>{42} + v_z \mathbf{e}<em>{43} + m_x \mathbf{e}</em>{32} + m_y \mathbf{e}<em>{31} + m_z \mathbf{e}</em>{12} )</td>
<td>( | \mathbf{L} |_\bullet = \sqrt{m_x^2 + m_y^2 + m_z^2} )</td>
<td>( | \mathbf{L} |_\circ = \sqrt{v_x^2 + v_y^2 + v_z^2} )</td>
</tr>
<tr>
<td>Plane</td>
<td>( \mathbf{f} = f_x \mathbf{e}<em>{234} + f_y \mathbf{e}</em>{314} + f_z \mathbf{e}<em>{124} + f_w \mathbf{e}</em>{321} )</td>
<td>( | \mathbf{f} |_\bullet =</td>
<td>f_w</td>
</tr>
<tr>
<td>Motor</td>
<td>( \mathbf{Q} = r_x \mathbf{e}<em>{41} + r_y \mathbf{e}</em>{42} + r_z \mathbf{e}<em>{43} + r_w \mathbf{1} + u_x \mathbf{e}</em>{32} + u_y \mathbf{e}<em>{31} + u_z \mathbf{e}</em>{12} + u_w )</td>
<td>( | \mathbf{Q} |_\bullet = \sqrt{u_x^2 + u_y^2 + u_z^2 + u_w^2} )</td>
<td>( | \mathbf{Q} |_\circ = \sqrt{r_x^2 + r_y^2 + r_z^2 + r_w^2} )</td>
</tr>
<tr>
<td>Flector</td>
<td>( \mathbf{G} = s_x \mathbf{e}<em>1 + s_y \mathbf{e}<em>2 + s_z \mathbf{e}<em>3 + s_w \mathbf{e}<em>4 + h_x \mathbf{e}</em>{234} + h_y \mathbf{e}</em>{314} + h_z \mathbf{e}</em>{124} + h_w \mathbf{e}</em>{321} )</td>
<td>( | \mathbf{G} |_\bullet = \sqrt{s_x^2 + s_y^2 + s_z^2 + s_w^2} )</td>
<td>( | \mathbf{G} |_\circ = \sqrt{h_x^2 + h_y^2 + h_z^2 + h_w^2} )</td>
</tr>
</tbody>
</table>
Unitization

- An object is unitized when its weight norm is 1
- This happens when the coefficients satisfy the following conditions

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
<th>Unitization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>( p = p_1 e_1 + p_2 e_2 + p_3 e_3 + p_4 e_4 )</td>
<td>( p_w^2 = 1 )</td>
</tr>
<tr>
<td>Line</td>
<td>( L = v_1 e_{41} + v_2 e_{42} + v_3 e_{43} + m_x e_{23} + m_y e_{31} + m_z e_{12} )</td>
<td>( v_1^2 + v_2^2 + v_3^2 = 1 )</td>
</tr>
<tr>
<td>Plane</td>
<td>( f = f_x e_{234} + f_y e_{314} + f_z e_{124} + f_w e_{321} )</td>
<td>( f_x^2 + f_y^2 + f_z^2 = 1 )</td>
</tr>
<tr>
<td>Motor</td>
<td>( Q = r_x e_{41} + r_y e_{42} + r_z e_{43} + u_x e_{23} + u_y e_{31} + u_z e_{12} + u_w )</td>
<td>( r_x^2 + r_y^2 + r_z^2 + u_x^2 + u_y^2 + u_z^2 + u_w^2 = 1 )</td>
</tr>
<tr>
<td>Flector</td>
<td>( G = s_x e_1 + s_y e_2 + s_z e_3 + s_w e_4 + h_x e_{234} + h_y e_{314} + h_z e_{124} + h_w e_{321} )</td>
<td>( s_x^2 + s_y^2 + s_z^2 + s_w^2 + h_x^2 + h_y^2 + h_z^2 + h_w^2 = 1 )</td>
</tr>
</tbody>
</table>
Homogeneous Magnitude

● What do the norms represent?

● We add them together and get a scalar/antiscalar pair \( x_1 + y_1 \)

● This is a homogeneous magnitude that has a bulk and a weight and can also be unitized!
# Homogeneous Magnitude

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
<th>Homogeneous Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>$\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4$</td>
<td>$</td>
</tr>
<tr>
<td>Line</td>
<td>$\mathbf{L} = v_x \mathbf{e}<em>{41} + v_y \mathbf{e}</em>{42} + v_z \mathbf{e}<em>{43} + m_x \mathbf{e}</em>{23} + m_y \mathbf{e}<em>{31} + m_z \mathbf{e}</em>{12}$</td>
<td>$</td>
</tr>
<tr>
<td>Plane</td>
<td>$\mathbf{f} = f_x \mathbf{e}<em>{234} + f_y \mathbf{e}</em>{314} + f_z \mathbf{e}<em>{124} + f_w \mathbf{e}</em>{321}$</td>
<td>$</td>
</tr>
<tr>
<td>Motor</td>
<td>$\mathbf{Q} = r_x \mathbf{e}<em>{41} + r_y \mathbf{e}</em>{42} + r_z \mathbf{e}<em>{43} + r_w \mathbf{1} + u_x \mathbf{e}</em>{23} + u_y \mathbf{e}<em>{31} + u_z \mathbf{e}</em>{12} + u_w$</td>
<td>$</td>
</tr>
<tr>
<td>Flector</td>
<td>$\mathbf{G} = s_x \mathbf{e}<em>1 + s_y \mathbf{e}<em>2 + s_z \mathbf{e}<em>3 + s_w \mathbf{e}<em>4 + h_x \mathbf{e}</em>{234} + h_y \mathbf{e}</em>{314} + h_z \mathbf{e}</em>{124} + h_w \mathbf{e}</em>{321}$</td>
<td>$</td>
</tr>
</tbody>
</table>
Geometric Norm

- The geometric norm is produced by unitizing the homogeneous magnitude so that its weight (the antiscalar part) is just $1$

- This gives us a concrete measurement of Euclidean distance
# Geometric Norm

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
<th>Geometric Norm</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>$p = p_x e_1 + p_y e_2 + p_z e_3 + p_w e_4$</td>
<td>$|p| = \sqrt{p_x^2 + p_y^2 + p_z^2} / |p_w|$</td>
<td>Distance from the origin to the point $p$. Half the distance that the origin is moved by the flector $p$.</td>
</tr>
<tr>
<td>Line</td>
<td>$L = v_x e_{a1} + v_y e_{a2} + v_z e_{a3} + m_x e_{b23} + m_y e_{b31} + m_z e_{b12}$</td>
<td>$|L| = \sqrt{m_x^2 + m_y^2 + m_z^2} / \sqrt{v_x^2 + v_y^2 + v_z^2}$</td>
<td>Perpendicular distance from the origin to the line $L$. Half the distance that the origin is moved by the motor $L$.</td>
</tr>
<tr>
<td>Plane</td>
<td>$f = f_x e_{234} + f_y e_{314} + f_z e_{124} + f_w e_{321}$</td>
<td>$|f| = \frac{</td>
<td>f_w</td>
</tr>
<tr>
<td>Motor</td>
<td>$Q = r_x e_{a1} + r_y e_{a2} + r_z e_{a3} + r_w 1 + u_x e_{b23} + u_y e_{b31} + u_z e_{b12} + u_w$</td>
<td>$|Q| = \sqrt{u_x^2 + u_y^2 + u_z^2 + u_w^2} / \sqrt{r_x^2 + r_y^2 + r_z^2 + r_w^2}$</td>
<td>Half the distance that the origin is moved by the motor $Q$.</td>
</tr>
<tr>
<td>Flector</td>
<td>$G = s_x e_1 + s_y e_2 + s_z e_3 + s_w e_4 + h_x e_{234} + h_y e_{314} + h_z e_{124} + h_w e_{321}$</td>
<td>$|G| = \sqrt{s_x^2 + s_y^2 + s_z^2 + h_w^2} / \sqrt{h_x^2 + h_y^2 + h_z^2 + s_w^2}$</td>
<td>Half the distance that the origin is moved by the flector $G$.</td>
</tr>
</tbody>
</table>
Commutators

- Four different commutators
- Combining addition or subtraction with geometric product or antiproduct

\[
\begin{align*}
[a, b]^\wedge_- &= \frac{1}{2} (a \wedge \tilde{b} - b \wedge \tilde{a}) & [a, b]^\vee_- &= \frac{1}{2} (a \vee \tilde{b} - b \vee \tilde{a}) \\
[a, b]^\wedge_+ &= \frac{1}{2} (a \wedge \tilde{b} + b \wedge \tilde{a}) & [a, b]^\vee_+ &= \frac{1}{2} (a \vee \tilde{b} + b \vee \tilde{a})
\end{align*}
\]
Commutators

- All of the join and meet operations can be done with commutators, but there’s more...

- A commutator can construct the line between two lines

- Commutators also give Euclidean distances between different objects
Line Between Two Lines

\[
[K, L]^\nu = (v_y w_z - v_z w_y) e_{41} + (v_z w_x - v_x w_z) e_{42} + (v_x w_y - v_y w_x) e_{43} \\
+ (v_y n_z - v_z n_y + m_y w_z - m_z w_y) e_{23} + (v_z n_x - v_x n_z + m_z w_x - m_x w_z) e_{31} + (v_x n_y - v_y n_x + m_x w_y - m_y w_x) e_{12}
\]
## Euclidean Distances

<table>
<thead>
<tr>
<th>Formula</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|\mathbf{p}, \mathbf{q}|_\circ = \sqrt{(q_x p_w - p_x q_w)^2 + (q_y p_w - p_y q_w)^2 + (q_z p_w - p_z q_w)^2} /</td>
<td>p_w q_w</td>
</tr>
<tr>
<td>$|\mathbf{p}, \mathbf{L}|_\circ = \sqrt{(v_z p_w - v_y p_x + m_z p_w)^2 + (v_z p_w - v_y p_x + m_z p_w)^2 + (v_z p_w - v_y p_x + m_z p_w)^2} /</td>
<td>p_w</td>
</tr>
<tr>
<td>$|\mathbf{p}, \mathbf{f}|_\circ = \sqrt{p_x f_z + p_y f_z + p_z f_z + p_w f_w}$</td>
<td>Perpendicular distance between point $\mathbf{p}$ and plane $\mathbf{f}$.</td>
</tr>
<tr>
<td>$|\mathbf{K}, \mathbf{L}|_\circ = \sqrt{(v_y n_z + v_z n_y + v_z n_z + w_z m_z + w_z m_z)^2 + (v_z w_y - v_y w_z)^2 + (v_z w_y - v_y w_z)^2}$</td>
<td>Perpendicular distance between lines $\mathbf{K}$ and $\mathbf{L}$.</td>
</tr>
</tbody>
</table>
Rotation

- Reflect through plane \( f \) and then plane \( g \)
  
  \[
  g \triangleright f = (f_y g_z - f_z g_y) e_{41} + (f_z g_x - f_x g_z) e_{42} + (f_x g_y - f_y g_x) e_{43} \\
  + (f_w g_x - f_x g_w) e_{23} + (f_w g_y - f_y g_w) e_{31} + (f_w g_z - f_z g_w) e_{12} \\
  + (f_x g_x + f_y g_y + f_z g_z) \mathbb{1}
  \]

- Contains line where planes intersect
- Also contains angle information
Rotation

- Rotate about line $L$ through angle $2\phi$

$R = L \sin \phi + 1 \cos \phi$
Translation

- Parallel planes intersect at a line at infinity
- And angle between them is zero

\[ T = t_x e_{23} + t_y e_{31} + t_z e_{12} + 1 \]
Motor

- All proper 3D isometries can be described as a screw motion.
- A rotation about a line and a displacement along the same line.
- General form of motor:

\[ Q = L \sin \phi + 1 \cos \phi + (d \vee L) \cos \phi - d \sin \phi \]
## Motors from Geometries

<table>
<thead>
<tr>
<th>Motor</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{g} \vee \mathbf{f} = (f_3 g_3 - f_3 g_3) \mathbf{e}_4 + (f_3 g_3 - f_3 g_3) \mathbf{e}<em>5 + (f_3 g_3 - f_3 g_3) \mathbf{e}<em>6 + (f_3 g_3 - f_3 g_3) \mathbf{e}<em>7 + (f_3 g_3 - f_3 g_3) \mathbf{e}<em>8 + (f_3 g_3 - f_3 g_3) \mathbf{e}<em>9 + (f_3 g_3 - f_3 g_3) \mathbf{e}</em>{10} + (f_3 g_3 - f_3 g_3) \mathbf{e}</em>{11} + (f_3 g_3 - f_3 g_3) \mathbf{e}</em>{12} + (f_3 g_3 - f_3 g_3) \mathbf{e}</em>{13} + (f_3 g_3 - f_3 g_3) \mathbf{e}</em>{14} )</td>
<td>Rotation about the line where planes ( \mathbf{f} ) and ( \mathbf{g} ) intersect by twice the angle between them in the direction from ( \mathbf{f} ) to ( \mathbf{g} ).</td>
</tr>
<tr>
<td>( \mathbf{L} \vee \mathbf{K} = (v_1 w_1 - v_1 w_1) \mathbf{e}_4 + (v_1 w_1 - v_1 w_1) \mathbf{e}<em>5 + (v_1 w_1 - v_1 w_1) \mathbf{e}<em>6 + (v_1 w_1 - v_1 w_1) \mathbf{e}<em>7 + (v_1 w_1 - v_1 w_1) \mathbf{e}<em>8 + (v_1 w_1 - v_1 w_1) \mathbf{e}<em>9 + (v_1 w_1 - v_1 w_1) \mathbf{e}</em>{10} + (v_1 w_1 - v_1 w_1) \mathbf{e}</em>{11} + (v_1 w_1 - v_1 w_1) \mathbf{e}</em>{12} + (v_1 w_1 - v_1 w_1) \mathbf{e}</em>{13} + (v_1 w_1 - v_1 w_1) \mathbf{e}</em>{14} )</td>
<td>Rotation about the line containing the closest points on lines ( \mathbf{K} ) and ( \mathbf{L} ) by twice the angle between ( \mathbf{v} ) and ( \mathbf{w} ).</td>
</tr>
<tr>
<td>( \mathbf{L} = { \mathbf{v}</td>
<td>\mathbf{m} } \quad \mathbf{K} = { \mathbf{w}</td>
</tr>
<tr>
<td>( \mathbf{q} \vee \mathbf{p} = (p_3 + q_3 - q_3) \mathbf{e}<em>{23} + (p_3 + q_3 - q_3) \mathbf{e}</em>{24} + (p_3 + q_3 - q_3) \mathbf{e}<em>{25} + (p_3 + q_3 - q_3) \mathbf{e}</em>{26} + (p_3 + q_3 - q_3) \mathbf{e}<em>{27} + (p_3 + q_3 - q_3) \mathbf{e}</em>{28} )</td>
<td>Translation by twice the distance between points ( \mathbf{p} ) and ( \mathbf{q} ) in the direction from ( \mathbf{p} ) to ( \mathbf{q} ).</td>
</tr>
</tbody>
</table>
Motor to Matrix

● We eventually want to convert to a 4×4 matrix

● Not as efficient to compute sandwich products a bunch of times

● Let $\mathbf{M}$ be the 4×4 matrix that we would use to transform points
Motor to Matrix

- Define

\[
A = \begin{bmatrix}
1 - 2 \left( r_y^2 + r_z^2 \right) & 2r_y r_x & 2r_z r_x & 2(r_y u_z - r_z u_y) \\
2r_x r_y & 1 - 2 \left( r_z^2 + r_x^2 \right) & 2r_y r_z & 2(r_z u_x - r_x u_z) \\
2r_z r_x & 2r_y r_z & 1 - 2 \left( r_x^2 + r_y^2 \right) & 2(r_x u_y - r_y u_x) \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & -2r_z r_w & 2r_y r_w & 2(r_w u_x - r_x u_w) \\
2r_z r_w & 0 & -2r_x r_w & 2(r_w u_y - r_y u_w) \\
-2r_y r_w & 2r_x r_w & 0 & 2(r_w u_z - r_z u_w) \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

- Then \( M = A + B \) and \( M^{-1} = A - B \)
Motor Advantages

- Arbitrary rigid motion can be stored as 6 floats

\[
Q = r_x e_{41} + r_y e_{42} + r_z e_{43} + r_w 1 + u_x e_{23} + u_y e_{31} + u_z e_{12} + u_w
\]

Rotation

- To be unitized, rotation part has unit length
- Can flip sign to make \( r_w \geq 0 \)
- Then

\[
r_w = \sqrt{1 - r_x^2 - r_y^2 - r_z^2}
\]
Motor Advantages

- Geometric property requires
  \[ r_x u_x + r_y u_y + r_z u_z + r_w u_w = 0 \]

- Can solve for \( u_w \) when other 7 values known

- Not a coincidence that a rigid motion in 3D space has 6 degrees of freedom
Motor Advantages

- Extremely easy to invert: $Q^{-1} = \tilde{Q}$
- This just negates the 6 bivector components

- Easier to re-orthogonalize than $4 \times 4$ matrix
  - Unitize the weight part $r$
  - Subtract projection of bulk part $u$ onto weight part
Motor Interpolation

- Motors interpolate a lot better than matrices
- This is used for dual quaternion skinning
Motor Interpolation

- A motor can be expressed as an exponential with respect to the geometric antiproduct:

\[
Q = e^{(d+\varphi \mathbf{1}) \vee \mathbf{L}} = \cos_\vee (d + \varphi \mathbf{1}) + \sin_\vee (d + \varphi \mathbf{1}) \vee \mathbf{L}
\]

\[
Q = \mathbf{1} \cos \varphi - d \sin \varphi + (d \vee \mathbf{L}) \cos \varphi + \mathbf{L} \sin \varphi
\]

\[
Q = (v_x e_{41} + v_y e_{42} + v_z e_{43}) \sin \varphi + \mathbf{1} \cos \varphi - d \sin \varphi
\]

\[
+ (dv_x e_{23} + dv_y e_{31} + dv_z e_{12}) \cos \varphi
\]

\[
+ (m_x e_{23} + m_y e_{31} + m_z e_{12}) \sin \varphi
\]
Motor Interpolation

- The exponential form allows for high-quality interpolation, but requires a logarithm.

- In practice, linear interpolation and re-unitization are sufficient.
Dual Quaternion Skinning
Reflection and Inversion

- Planes and points as isometry operators
Transflection

- A plane and a direction

\[ H = s_x e_1 + s_y e_2 + s_z e_3 + h \]
Flector

● All improper 3D isometries can be described as a rotoreflection
● This is a rotation about a line and a reflection through a plane perpendicular to that line
● General form of a flector
Flector

\[ G = p \sin \phi + f \cos \phi \]
Flector to Matrix

- **Define**
  
  \[
  A = \begin{bmatrix}
  2(h_y^2 + h_z^2) - 1 & -2h_xh_y & -2h_zh_x & 2(s_x s_w - h_x h_w) \\
  -2h_xh_y & 2(h_z^2 + h_x^2) - 1 & -2h_yh_z & 2(s_y s_w - h_y h_w) \\
  -2h_zh_x & -2h_yh_z & 2(h_x^2 + h_y^2) - 1 & 2(s_z s_w - h_z h_w) \\
  0 & 0 & 0 & 1 
  \end{bmatrix}
  \]

  \[
  B = \begin{bmatrix}
  0 & 2h_z s_w & -2h_y s_w & 2(h_y s_z - h_z s_y) \\
  -2h_z s_w & 0 & 2h_x s_w & 2(h_z s_x - h_x s_z) \\
  2h_y s_w & -2h_x s_w & 0 & 2(h_x s_y - h_y s_x) \\
  0 & 0 & 0 & 0 
  \end{bmatrix}
  \]

- **Then** \( M = A + B \) and \( M^{-1} = A - B \)
More Information

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