Geometry and Motion in Projective Grassmann and Clifford Algebras

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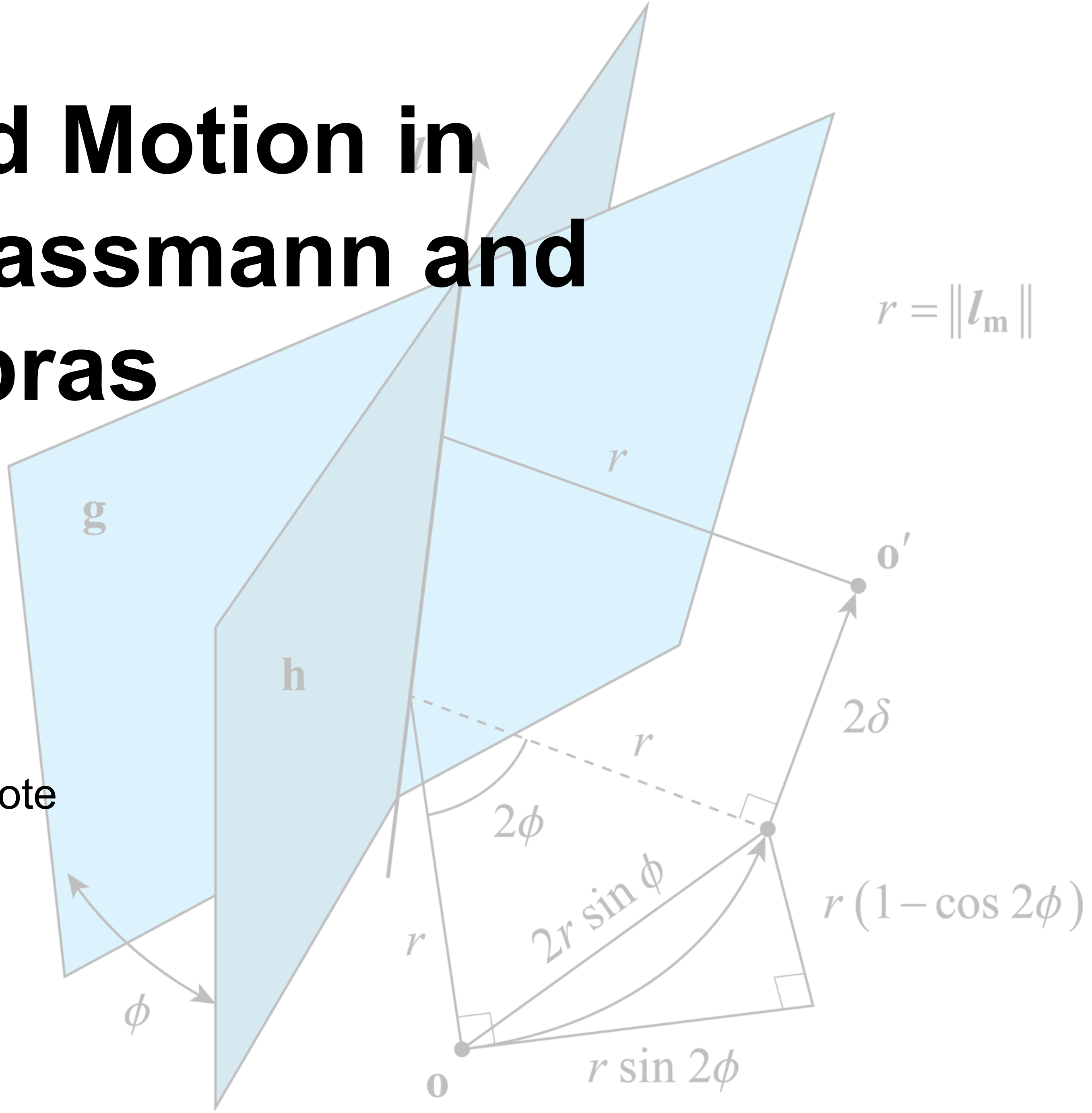
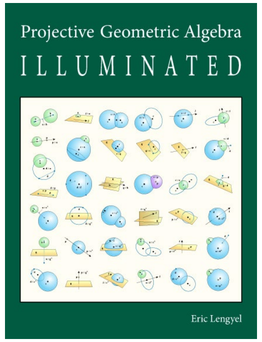
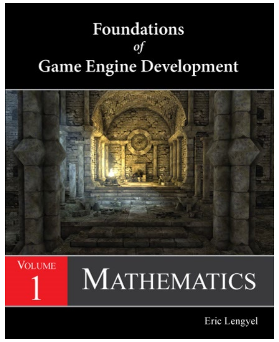
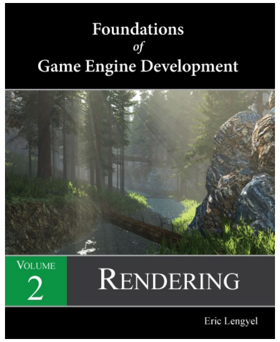


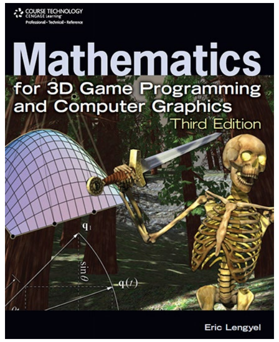
About the Speaker

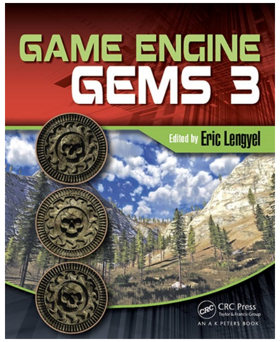
- Computer Scientist / Mathematician
- Working in industry since 1994
- Developing algebraic models for about 15 years
- Occasionally teaches computer graphics
- Writes books about math and real-time rendering

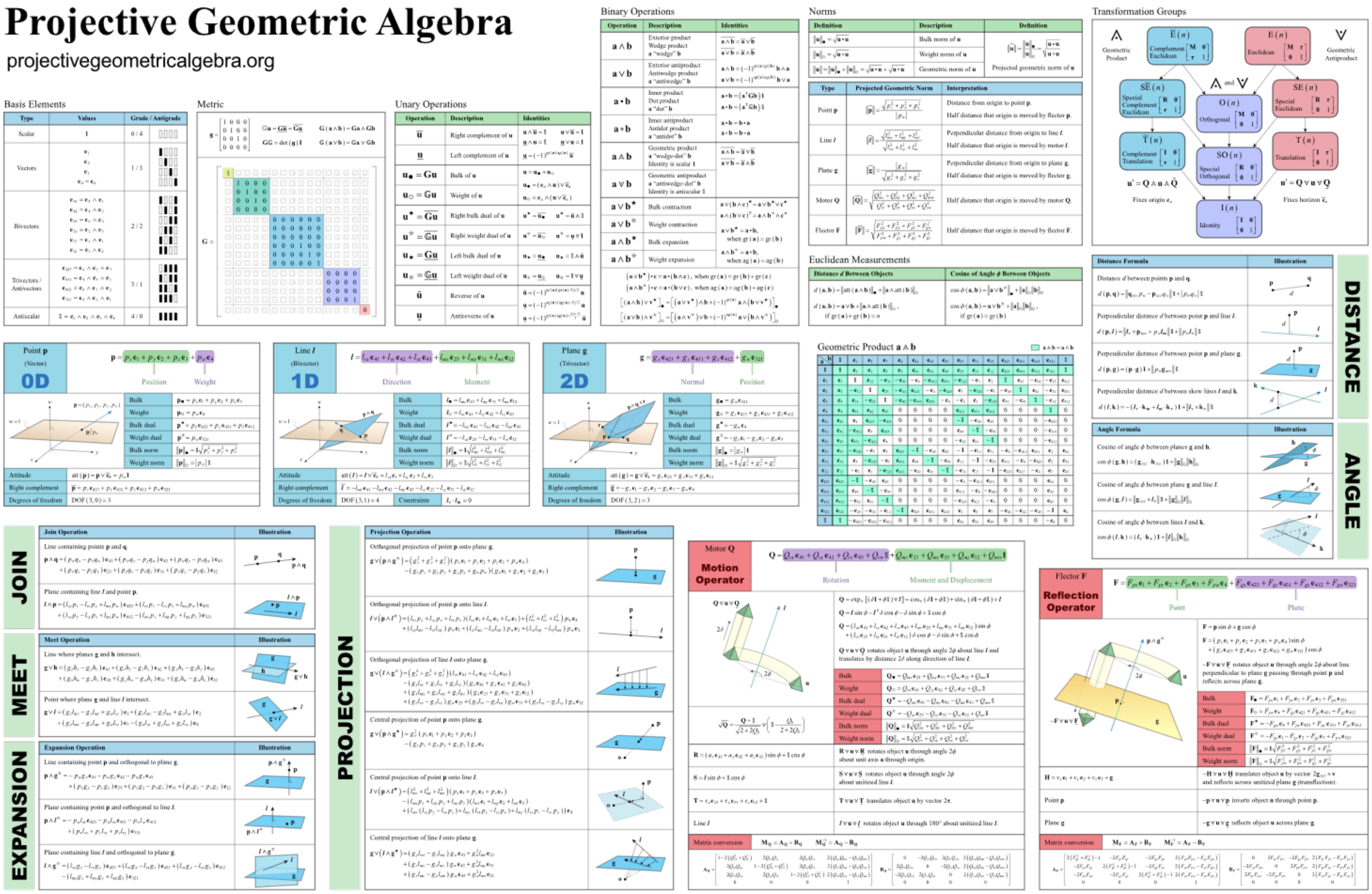












Projective Geometric Algebra

projectivegeometricalgebra.org

Basis Elements

Type	Values	Grade / Antigrade	
Scalar	1	0 / 4	□□□□
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4 = \mathbf{e}_w$	1 / 3	■□□□ □■□□ □□■□ □□□■
Bivectors	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$	2 / 2	■□□■ □■□■ □□■■ □■■□ ■□■□ ■■□□
Trivectors / Antivectors	$\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$	3 / 1	□■■■ ■□■■ ■■□■ ■■■□
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	■■■■

Metric

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{G}\mathbf{u} = \underline{\mathbb{G}\overline{\mathbf{u}}} = \overline{\mathbb{G}\underline{\mathbf{u}}} \qquad \mathbf{G}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{G}\mathbf{a} \wedge \mathbf{G}\mathbf{b}$$

$$\mathbf{G}\mathbb{G} = \det(\mathfrak{g})\,\mathbf{I} \qquad \mathbb{G}(\mathbf{a} \vee \mathbf{b}) = \mathbb{G}\mathbf{a} \vee \mathbb{G}\mathbf{b}$$

$$\mathbf{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square\\ \hline
\square & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square\\
\square & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square\\
\square & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square\\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square\\ \hline
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square\\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square\\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square\\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square\\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square\\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square\\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square\\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square\\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square\\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & \square\\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0
\end{array}\right]$$

Unary Operations

Operation	Description	Identities
$\overline{\mathbf{u}}$	Right complement of $\mathbf{u}$	$\mathbf{u} \wedge \overline{\mathbf{u}} = \mathbb{1}$ $\quad \mathbf{u} \vee \overline{\mathbf{u}} = 1$ $\underline{\mathbf{u}} \wedge \mathbf{u} = \mathbb{1}$ $\quad \underline{\mathbf{u}} \vee \mathbf{u} = 1$ $\underline{\mathbf{u}} = (-1)^{\mathrm{gr}(\mathbf{u})\,\mathrm{ag}(\mathbf{u})}\,\overline{\mathbf{u}}$
$\underline{\mathbf{u}}$	Left complement of $\mathbf{u}$	
$\mathbf{u}_{\bullet} = \mathbf{G}\mathbf{u}$	Bulk of $\mathbf{u}$	$\mathbf{u} = \mathbf{u}_{\bullet} + \mathbf{u}_{\circ}$ $\mathbf{u}_{\bullet} = (\mathbf{e}_4 \wedge \mathbf{u}) \vee \overline{\mathbf{e}}_4$ $\mathbf{u}_{\circ} = \mathbf{e}_4 \wedge (\mathbf{u} \vee \overline{\mathbf{e}}_4)$
$\mathbf{u}_{\circ} = \mathbb{G}\mathbf{u}$	Weight of $\mathbf{u}$	
$\mathbf{u}^{★} = \overline{\mathbf{G}\mathbf{u}}$	Right bulk dual of $\mathbf{u}$	$\mathbf{u}^{★} = \overline{\mathbf{u}_{\bullet}}$ $\quad \mathbf{u}^{★} = \tilde{\mathbf{u}} ⟑ \mathbb{1}$
$\mathbf{u}^{☆} = \overline{\mathbb{G}\mathbf{u}}$	Right weight dual of $\mathbf{u}$	$\mathbf{u}^{☆} = \overline{\mathbf{u}_{\circ}}$ $\quad \mathbf{u}^{☆} = \underset{\sim}{\mathbf{u}} ⟇ 1$
$\mathbf{u}_{★} = \underline{\mathbf{G}\mathbf{u}}$	Left bulk dual of $\mathbf{u}$	$\mathbf{u}_{★} = \underline{\mathbf{u}_{\bullet}}$ $\quad \mathbf{u}_{★} = \mathbb{1} ⟑ \tilde{\mathbf{u}}$
$\mathbf{u}_{☆} = \underline{\mathbb{G}\mathbf{u}}$	Left weight dual of $\mathbf{u}$	$\mathbf{u}_{☆} = \underline{\mathbf{u}_{\circ}}$ $\quad \mathbf{u}_{☆} = 1 ⟇ \underset{\sim}{\mathbf{u}}$
$\tilde{\mathbf{u}}$	Reverse of $\mathbf{u}$	$\tilde{\mathbf{u}} = (-1)^{\mathrm{gr}(\mathbf{u})(\mathrm{gr}(\mathbf{u})-1)/2}\,\mathbf{u}$ $\underset{\sim}{\mathbf{u}} = (-1)^{\mathrm{ag}(\mathbf{u})(\mathrm{ag}(\mathbf{u})-1)/2}\,\mathbf{u}$ $\underset{\sim}{\mathbf{u}} = (-1)^{\mathrm{gr}(\mathbf{u})\,\mathrm{ag}(\mathbf{u}) + \frac{n(n-1)}{2}}\,\tilde{\mathbf{u}}$
$\underset{\sim}{\mathbf{u}}$	Antireverse of $\mathbf{u}$	

Binary Operations

Operation	Description	Identities
$\mathbf{a} \wedge \mathbf{b}$	Exterior product Wedge product a "wedge" b	$\overline{\mathbf{a} \wedge \mathbf{b}} = \overline{\mathbf{a}} \vee \overline{\mathbf{b}}$ $\overline{\mathbf{a} \vee \mathbf{b}} = \overline{\mathbf{a}} \wedge \overline{\mathbf{b}}$ $\mathbf{a} \wedge \mathbf{b} = (-1)^{\mathrm{gr}(\mathbf{a})\,\mathrm{gr}(\mathbf{b})}\,\mathbf{b} \wedge \mathbf{a}$ $\mathbf{a} \vee \mathbf{b} = (-1)^{\mathrm{ag}(\mathbf{a})\,\mathrm{ag}(\mathbf{b})}\,\mathbf{b} \vee \mathbf{a}$
$\mathbf{a} \vee \mathbf{b}$	Exterior antiproduct Antiwedge product a "antiwedge" b	
$\mathbf{a} \bullet \mathbf{b}$	Inner product Dot product a "dot" b	$\mathbf{a} \bullet \mathbf{b} = (\mathbf{a}^{\mathrm{T}}\mathbf{G}\mathbf{b})\,1$ $\mathbf{a} \circ \mathbf{b} = (\mathbf{a}^{\mathrm{T}}\mathbb{G}\mathbf{b})\,\mathbb{1}$ $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ $\mathbf{a} \circ \mathbf{b} = \mathbf{b} \circ \mathbf{a}$
$\mathbf{a} \circ \mathbf{b}$	Inner antiproduct Antidot product a "antidot" b	
$\mathbf{a} ⟑ \mathbf{b}$	Geometric product a "wedge-dot" b Identity is scalar 1	$\overline{\mathbf{a} ⟑ \mathbf{b}} = \overline{\mathbf{a}} ⟇ \overline{\mathbf{b}}$ $\overline{\mathbf{a} ⟇ \mathbf{b}} = \overline{\mathbf{a}} ⟑ \overline{\mathbf{b}}$
$\mathbf{a} ⟇ \mathbf{b}$	Geometric antiproduct a "antiwedge-dot" b Identity is antiscalar $\mathbb{1}$	
$\mathbf{a} \vee \mathbf{b}^{★}$	Bulk contraction	$\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c})^{★} = \mathbf{a} \vee \mathbf{b}^{★} \vee \mathbf{c}^{★}$ $\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})^{☆} = \mathbf{a} \wedge \mathbf{b}^{☆} \wedge \mathbf{c}^{☆}$ $\mathbf{a} \vee \mathbf{b}^{★} = \mathbf{a} \bullet \mathbf{b}$, when $\mathrm{gr}(\mathbf{a}) = \mathrm{gr}(\mathbf{b})$ $\mathbf{a} \wedge \mathbf{b}^{☆} = \mathbf{a} \circ \mathbf{b}$, when $\mathrm{ag}(\mathbf{a}) = \mathrm{ag}(\mathbf{b})$
$\mathbf{a} \vee \mathbf{b}^{☆}$	Weight contraction	
$\mathbf{a} \wedge \mathbf{b}^{★}$	Bulk expansion	
$\mathbf{a} \wedge \mathbf{b}^{☆}$	Weight expansion	

$$(\mathbf{a} \vee \mathbf{b}^{★}) \bullet \mathbf{c} = \mathbf{a} \bullet (\mathbf{b} \wedge \mathbf{c}),\ \text{when } \mathrm{gr}(\mathbf{a}) = \mathrm{gr}(\mathbf{b}) + \mathrm{gr}(\mathbf{c})$$

$$(\mathbf{a} \wedge \mathbf{b}^{☆}) \circ \mathbf{c} = \mathbf{a} \circ (\mathbf{b} \vee \mathbf{c}),\ \text{when } \mathrm{ag}(\mathbf{a}) = \mathrm{ag}(\mathbf{b}) + \mathrm{ag}(\mathbf{c})$$

$$\left[(\mathbf{a} \wedge \mathbf{b}) \vee \mathbf{v}^{★}\right]_{\bullet} = \left[(\mathbf{a} \vee \mathbf{v}^{★}) \wedge \mathbf{b} + (-1)^{\mathrm{gr}(\mathbf{a})}\,\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{v}^{★})\right]_{\bullet}$$

$$\left[(\mathbf{a} \vee \mathbf{b}) \wedge \mathbf{v}^{☆}\right]_{\circ} = \left[(\mathbf{a} \wedge \mathbf{v}^{☆}) \vee \mathbf{b} + (-1)^{\mathrm{ag}(\mathbf{a})}\,\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{v}^{☆})\right]_{\circ}$$

Norms

Definition	Description	Definition
$\left\Vert\mathbf{u}\right\Vert_{\bullet} = \sqrt{\mathbf{u} \bullet \mathbf{u}}$	Bulk norm of $\mathbf{u}$	$\widehat{\left\Vert\mathbf{u}\right\Vert} = \dfrac{\left\Vert\mathbf{u}\right\Vert_{\bullet}}{\left\Vert\mathbf{u}\right\Vert_{\circ}} = \sqrt{\dfrac{\mathbf{u} \bullet \mathbf{u}}{\mathbf{u} \circ \mathbf{u}}}$ Projected geometric norm of $\mathbf{u}$
$\left\Vert\mathbf{u}\right\Vert_{\circ} = \sqrt{\mathbf{u} \circ \mathbf{u}}$	Weight norm of $\mathbf{u}$	
$\left\Vert\mathbf{u}\right\Vert = \left\Vert\mathbf{u}\right\Vert_{\bullet} + \left\Vert\mathbf{u}\right\Vert_{\circ} = \sqrt{\mathbf{u} \bullet \mathbf{u}} + \sqrt{\mathbf{u} \circ \mathbf{u}}$	Geometric norm of $\mathbf{u}$	

Type	Projected Geometric Norm	Interpretation
Point $\mathbf{p}$	$\widehat{\left\Vert\mathbf{p}\right\Vert} = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from origin to point $\mathbf{p}$. Half distance that origin is moved by flector $\mathbf{p}$.
Line $\boldsymbol{l}$	$\widehat{\left\Vert\boldsymbol{l}\right\Vert} = \dfrac{\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}}{\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$	Perpendicular distance from origin to line $\boldsymbol{l}$. Half distance that origin is moved by motor $\boldsymbol{l}$.
Plane $\mathbf{g}$	$\widehat{\left\Vert\mathbf{g}\right\Vert} = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$	Perpendicular distance from origin to plane $\mathbf{g}$. Half distance that origin is moved by flector $\mathbf{g}$.
Motor $\mathbf{Q}$	$\widehat{\left\Vert\mathbf{Q}\right\Vert} = \sqrt{\dfrac{Q_{mx}^2 + Q_{my}^2 + Q_{mz}^2 + Q_{mw}^2}{Q_{vx}^2 + Q_{vy}^2 + Q_{vz}^2 + Q_{vw}^2}}$	Half distance that origin is moved by motor $\mathbf{Q}$.
Flector $\mathbf{F}$	$\widehat{\left\Vert\mathbf{F}\right\Vert} = \sqrt{\dfrac{F_{px}^2 + F_{py}^2 + F_{pz}^2 + F_{gw}^2}{F_{pw}^2 + F_{gx}^2 + F_{gy}^2 + F_{gz}^2}}$	Half distance that origin is moved by flector $\mathbf{F}$.

Transformation Groups

Euclidean Measurements

Distance d Between Objects	Cosine of Angle ϕ Between Objects
$d(\mathbf{a}, \mathbf{b}) = \left\Vert\mathrm{att}(\mathbf{a} \wedge \mathbf{b})\right\Vert_{\bullet} + \left\Vert\mathbf{a} \wedge \mathrm{att}(\mathbf{b})\right\Vert_{\circ}$	$\cos\phi(\mathbf{a}, \mathbf{b}) = \left\Vert\mathbf{a} \vee \mathbf{b}^{☆}\right\Vert_{\bullet} + \left\Vert\mathbf{a}\right\Vert_{\circ}\left\Vert\mathbf{b}\right\Vert_{\circ}$
$d(\mathbf{a}, \mathbf{b}) = \mathbf{a} \vee \mathbf{b} + \left\Vert\mathbf{a} \wedge \mathrm{att}(\mathbf{b})\right\Vert_{\circ}$, if $\mathrm{gr}(\mathbf{a}) + \mathrm{gr}(\mathbf{b}) = n$	$\cos\phi(\mathbf{a}, \mathbf{b}) = \mathbf{a} \vee \mathbf{b}^{☆} + \left\Vert\mathbf{a}\right\Vert_{\circ}\left\Vert\mathbf{b}\right\Vert_{\circ}$, if $\mathrm{gr}(\mathbf{a}) = \mathrm{gr}(\mathbf{b})$

Geometric Product $\mathbf{a} ⟑ \mathbf{b}$

🟩 $\mathbf{a} ⟑ \mathbf{b} = \mathbf{a} \wedge \mathbf{b}$

a \ b	1	e_1	e_2	e_3	e_4	e_{41}	e_{42}	e_{43}	e_{23}	e_{31}	e_{12}	e_{423}	e_{431}	e_{412}	e_{321}	$\mathbb{1}$
1	1	e_1	e_2	e_3	e_4	e_{41}	e_{42}	e_{43}	e_{23}	e_{31}	e_{12}	e_{423}	e_{431}	e_{412}	e_{321}	$\mathbb{1}$
e_1	e_1	1	e_{12}	$-e_{31}$	$-e_{41}$	$-e_4$	$-e_{412}$	e_{431}	$-e_{321}$	$-e_3$	e_2	$\mathbb{1}$	e_{43}	$-e_{42}$	$-e_{23}$	e_{423}
e_2	e_2	$-e_{12}$	1	e_{23}	$-e_{42}$	e_{412}	$-e_4$	$-e_{423}$	e_3	$-e_{321}$	$-e_1$	$-e_{43}$	$\mathbb{1}$	e_{41}	$-e_{31}$	e_{431}
e_3	e_3	e_{31}	$-e_{23}$	1	$-e_{43}$	$-e_{431}$	e_{423}	$-e_4$	$-e_2$	e_1	$-e_{321}$	e_{42}	$-e_{41}$	$\mathbb{1}$	$-e_{12}$	e_{412}
e_4	e_4	e_{41}	e_{42}	e_{43}	0	0	0	0	e_{423}	e_{431}	e_{412}	0	0	0	$\mathbb{1}$	0
e_{41}	e_{41}	e_4	e_{412}	$-e_{431}$	0	0	0	0	$-\mathbb{1}$	$-e_{43}$	e_{42}	0	0	0	$-e_{423}$	0
e_{42}	e_{42}	$-e_{412}$	e_4	e_{423}	0	0	0	0	e_{43}	$-\mathbb{1}$	$-e_{41}$	0	0	0	$-e_{431}$	0
e_{43}	e_{43}	e_{431}	$-e_{423}$	e_4	0	0	0	0	$-e_{42}$	e_{41}	$-\mathbb{1}$	0	0	0	$-e_{412}$	0
e_{23}	e_{23}	$-e_{321}$	$-e_3$	e_2	e_{423}	$-\mathbb{1}$	$-e_{43}$	e_{42}	-1	$-e_{12}$	e_{31}	$-e_4$	$-e_{412}$	e_{431}	e_1	e_{41}
e_{31}	e_{31}	e_3	$-e_{321}$	$-e_1$	e_{431}	e_{43}	$-\mathbb{1}$	$-e_{41}$	e_{12}	-1	$-e_{23}$	e_{412}	$-e_4$	$-e_{423}$	e_2	e_{42}
e_{12}	e_{12}	$-e_2$	e_1	$-e_{321}$	e_{412}	$-e_{42}$	e_{41}	$-\mathbb{1}$	$-e_{31}$	e_{23}	-1	$-e_{431}$	e_{423}	$-e_4$	e_3	e_{43}
e_{423}	e_{423}	$-\mathbb{1}$	$-e_{43}$	e_{42}	0	0	0	0	$-e_4$	$-e_{412}$	e_{431}	0	0	0	e_{41}	0
e_{431}	e_{431}	e_{43}	$-\mathbb{1}$	$-e_{41}$	0	0	0	0	e_{412}	$-e_4$	$-e_{423}$	0	0	0	e_{42}	0
e_{412}	e_{412}	$-e_{42}$	e_{41}	$-\mathbb{1}$	0	0	0	0	$-e_{431}$	e_{423}	$-e_4$	0	0	0	e_{43}	0
e_{321}	e_{321}	$-e_{23}$	$-e_{31}$	$-e_{12}$	$-\mathbb{1}$	e_{423}	e_{431}	e_{412}	e_1	e_2	e_3	$-e_{41}$	$-e_{42}$	$-e_{43}$	-1	e_4
$\mathbb{1}$	$\mathbb{1}$	$-e_{423}$	$-e_{431}$	$-e_{412}$	0	0	0	0	e_{41}	e_{42}	e_{43}	0	0	0	$-e_4$	0

Distance Formula	Illustration
Distance d between points $\mathbf{p}$ and $\mathbf{q}$. $d(\mathbf{p}, \mathbf{q}) = \left\Vert \mathbf{q}_{xyz}\,p_w - \mathbf{p}_{xyz}\,q_w \right\Vert 1 + \lvert p_w q_w \rvert \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and line $\boldsymbol{l}$. $d(\mathbf{p}, \boldsymbol{l}) = \left\Vert \boldsymbol{l}_v \times \mathbf{p}_{xyz} + p_w \boldsymbol{l}_m \right\Vert 1 + \left\Vert p_w \boldsymbol{l}_v \right\Vert \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and plane $\mathbf{g}$. $d(\mathbf{p}, \mathbf{g}) = (\mathbf{p} \cdot \mathbf{g})\,1 + \left\Vert p_w \mathbf{g}_{xyz} \right\Vert \mathbb{1}$	
Perpendicular distance d between skew lines $\boldsymbol{l}$ and $\mathbf{k}$. $d(\boldsymbol{l}, \mathbf{k}) = -(\boldsymbol{l}_v \cdot \mathbf{k}_m + \boldsymbol{l}_m \cdot \mathbf{k}_v)\,1 + \left\Vert \boldsymbol{l}_v \times \mathbf{k}_v \right\Vert \mathbb{1}$	

DISTANCE

Angle Formula	Illustration
Cosine of angle ϕ between planes $\mathbf{g}$ and $\mathbf{h}$. $\cos\phi(\mathbf{g}, \mathbf{h}) = (\mathbf{g}_{xyz} \cdot \mathbf{h}_{xyz})\,1 + \left\Vert\mathbf{g}\right\Vert_{\circ}\left\Vert\mathbf{h}\right\Vert_{\circ}$	
Cosine of angle ϕ between plane $\mathbf{g}$ and line $\boldsymbol{l}$. $\cos\phi(\mathbf{g}, \boldsymbol{l}) = \left\Vert \mathbf{g}_{xyz} \times \boldsymbol{l}_v \right\Vert 1 + \left\Vert\mathbf{g}\right\Vert_{\circ}\left\Vert\boldsymbol{l}\right\Vert_{\circ}$	
Cosine of angle ϕ between lines $\boldsymbol{l}$ and $\mathbf{k}$. $\cos\phi(\boldsymbol{l}, \mathbf{k}) = (\boldsymbol{l}_v \cdot \mathbf{k}_v)\,1 + \left\Vert\boldsymbol{l}\right\Vert_{\circ}\left\Vert\mathbf{k}\right\Vert_{\circ}$	

ANGLE

Point $\mathbf{p}$ (Vector) 0D

$$\mathbf{p} = \underbrace{p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3}_{\text{Position}} + \underbrace{p_w \mathbf{e}_4}_{\text{Weight}}$$

Bulk	$\mathbf{p}_{\bullet} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3$
Weight	$\mathbf{p}_{\circ} = p_w \mathbf{e}_4$
Bulk dual	$\mathbf{p}^{★} = p_x \mathbf{e}_{423} + p_y \mathbf{e}_{431} + p_z \mathbf{e}_{412}$
Weight dual	$\mathbf{p}^{☆} = p_w \mathbf{e}_{321}$
Bulk norm	$\left\Vert\mathbf{p}\right\Vert_{\bullet} = 1\sqrt{p_x^2 + p_y^2 + p_z^2}$
Weight norm	$\left\Vert\mathbf{p}\right\Vert_{\circ} = \lvert p_w \rvert \mathbb{1}$
Attitude	$\mathrm{att}(\mathbf{p}) = \mathbf{p} \vee \overline{\mathbf{e}}_4 = p_w 1$
Right complement	$\overline{\mathbf{p}} = p_x \mathbf{e}_{423} + p_y \mathbf{e}_{431} + p_z \mathbf{e}_{412} + p_w \mathbf{e}_{321}$
Degrees of freedom	$\mathrm{DOF}(3,0) = 3$

Line $\boldsymbol{l}$ (Bivector) 1D

$$\boldsymbol{l} = \underbrace{l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx} \mathbf{e}_{23} + l_{my} \mathbf{e}_{31} + l_{mz} \mathbf{e}_{12}}_{\text{Moment}}$$

Bulk	$\boldsymbol{l}_{\bullet} = l_{mx} \mathbf{e}_{23} + l_{my} \mathbf{e}_{31} + l_{mz} \mathbf{e}_{12}$		
Weight	$\boldsymbol{l}_{\circ} = l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43}$		
Bulk dual	$\boldsymbol{l}^{★} = -l_{mx} \mathbf{e}_{41} - l_{my} \mathbf{e}_{42} - l_{mz} \mathbf{e}_{43}$		
Weight dual	$\boldsymbol{l}^{☆} = -l_{vx} \mathbf{e}_{23} - l_{vy} \mathbf{e}_{31} - l_{vz} \mathbf{e}_{12}$		
Bulk norm	$\left\Vert\boldsymbol{l}\right\Vert_{\bullet} = 1\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$		
Weight norm	$\left\Vert\boldsymbol{l}\right\Vert_{\circ} = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$		
Attitude	$\mathrm{att}(\boldsymbol{l}) = \boldsymbol{l} \vee \overline{\mathbf{e}}_4 = l_{vx} \mathbf{e}_1 + l_{vy} \mathbf{e}_2 + l_{vz} \mathbf{e}_3$		
Right complement	$\overline{\boldsymbol{l}} = -l_{mx} \mathbf{e}_{41} - l_{my} \mathbf{e}_{42} - l_{mz} \mathbf{e}_{43} - l_{vx} \mathbf{e}_{23} - l_{vy} \mathbf{e}_{31} - l_{vz} \mathbf{e}_{12}$		
Degrees of freedom	$\mathrm{DOF}(3,1) = 4$	Constraints	$\boldsymbol{l}_v \cdot \boldsymbol{l}_m = 0$

Plane $\mathbf{g}$ (Trivector) 2D

$$\mathbf{g} = \underbrace{g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w \mathbf{e}_{321}}_{\text{Position}}$$

Bulk	$\mathbf{g}_{\bullet} = g_w \mathbf{e}_{321}$
Weight	$\mathbf{g}_{\circ} = g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}$
Bulk dual	$\mathbf{g}^{★} = -g_w \mathbf{e}_4$
Weight dual	$\mathbf{g}^{☆} = -g_x \mathbf{e}_1 - g_y \mathbf{e}_2 - g_z \mathbf{e}_3$
Bulk norm	$\left\Vert\mathbf{g}\right\Vert_{\bullet} = \lvert g_w \rvert 1$
Weight norm	$\left\Vert\mathbf{g}\right\Vert_{\circ} = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$
Attitude	$\mathrm{att}(\mathbf{g}) = \mathbf{g} \vee \overline{\mathbf{e}}_4 = g_x \mathbf{e}_{23} + g_y \mathbf{e}_{31} + g_z \mathbf{e}_{12}$
Right complement	$\overline{\mathbf{g}} = -g_x \mathbf{e}_1 - g_y \mathbf{e}_2 - g_z \mathbf{e}_3 - g_w \mathbf{e}_4$
Degrees of freedom	$\mathrm{DOF}(3,2) = 3$

JOIN

Join Operation	Illustration
Line containing points $\mathbf{p}$ and $\mathbf{q}$. $\mathbf{p} \wedge \mathbf{q} = (p_w q_x - p_x q_w)\mathbf{e}_{41} + (p_w q_y - p_y q_w)\mathbf{e}_{42} + (p_w q_z - p_z q_w)\mathbf{e}_{43}$ $+ (p_y q_z - p_z q_y)\mathbf{e}_{23} + (p_z q_x - p_x q_z)\mathbf{e}_{31} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$	
Plane containing line $\boldsymbol{l}$ and point $\mathbf{p}$. $\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx} p_w)\mathbf{e}_{423} + (l_{vz} p_x - l_{vx} p_z + l_{my} p_w)\mathbf{e}_{431}$ $+ (l_{vx} p_y - l_{vy} p_x + l_{mz} p_w)\mathbf{e}_{412} - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\mathbf{e}_{321}$	

MEET

Meet Operation	Illustration
Line where planes $\mathbf{g}$ and $\mathbf{h}$ intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\mathbf{e}_{41} + (g_x h_z - g_z h_x)\mathbf{e}_{42} + (g_y h_x - g_x h_y)\mathbf{e}_{43}$ $+ (g_x h_w - g_w h_x)\mathbf{e}_{23} + (g_y h_w - g_w h_y)\mathbf{e}_{31} + (g_z h_w - g_w h_z)\mathbf{e}_{12}$	
Point where plane $\mathbf{g}$ and line $\boldsymbol{l}$ intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\mathbf{e}_1 + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\mathbf{e}_2$ $+ (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\mathbf{e}_3 - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\mathbf{e}_4$	

EXPANSION

Expansion Operation	Illustration
Line containing point $\mathbf{p}$ and orthogonal to plane $\mathbf{g}$. $\mathbf{p} \wedge \mathbf{g}^{☆} = -p_w g_x \mathbf{e}_{41} - p_w g_y \mathbf{e}_{42} - p_w g_z \mathbf{e}_{43}$ $+ (p_z g_y - p_y g_z)\mathbf{e}_{23} + (p_x g_z - p_z g_x)\mathbf{e}_{31} + (p_y g_x - p_x g_y)\mathbf{e}_{12}$	
Plane containing point $\mathbf{p}$ and orthogonal to line $\boldsymbol{l}$. $\mathbf{p} \wedge \boldsymbol{l}^{☆} = -p_w l_{vx} \mathbf{e}_{423} - p_w l_{vy} \mathbf{e}_{431} - p_w l_{vz} \mathbf{e}_{412}$ $+ (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\mathbf{e}_{321}$	
Plane containing line $\boldsymbol{l}$ and orthogonal to plane $\mathbf{g}$. $\boldsymbol{l} \wedge \mathbf{g}^{☆} = (l_{vy} g_z - l_{vz} g_y)\mathbf{e}_{423} + (l_{vz} g_x - l_{vx} g_z)\mathbf{e}_{431} + (l_{vx} g_y - l_{vy} g_x)\mathbf{e}_{412}$ $- (l_{mx} g_x + l_{my} g_y + l_{mz} g_z)\mathbf{e}_{321}$	

PROJECTION

Projection Operation	Illustration
Orthogonal projection of point $\mathbf{p}$ onto plane $\mathbf{g}$. $\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{☆}) = (g_x^2 + g_y^2 + g_z^2)(p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4)$ $- (g_x p_x + g_y p_y + g_z p_z + g_w p_w)(g_x \mathbf{e}_1 + g_y \mathbf{e}_2 + g_z \mathbf{e}_3)$	
Orthogonal projection of point $\mathbf{p}$ onto line $\boldsymbol{l}$. $\boldsymbol{l} \vee (\mathbf{p} \wedge \boldsymbol{l}^{☆}) = (l_{vx} p_x + l_{vy} p_y + l_{vz} p_z)(l_{vx} \mathbf{e}_1 + l_{vy} \mathbf{e}_2 + l_{vz} \mathbf{e}_3) + (l_{vx}^2 + l_{vy}^2 + l_{vz}^2)\,p_w \mathbf{e}_4$ $+ (l_{vy} l_{mz} - l_{vz} l_{my})\,p_w \mathbf{e}_1 + (l_{vz} l_{mx} - l_{vx} l_{mz})\,p_w \mathbf{e}_2 + (l_{vx} l_{my} - l_{vy} l_{mx})\,p_w \mathbf{e}_3$	
Orthogonal projection of line $\boldsymbol{l}$ onto plane $\mathbf{g}$. $\mathbf{g} \vee (\boldsymbol{l} \wedge \mathbf{g}^{☆}) = (g_x^2 + g_y^2 + g_z^2)(l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43})$ $- (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})(g_x \mathbf{e}_{41} + g_y \mathbf{e}_{42} + g_z \mathbf{e}_{43})$ $+ (g_x l_{mx} + g_y l_{my} + g_z l_{mz})(g_x \mathbf{e}_{23} + g_y \mathbf{e}_{31} + g_z \mathbf{e}_{12})$ $+ (g_z l_{vy} - g_y l_{vz})\,g_w \mathbf{e}_{23} + (g_x l_{vz} - g_z l_{vx})\,g_w \mathbf{e}_{31} + (g_y l_{vx} - g_x l_{vy})\,g_w \mathbf{e}_{12}$	
Central projection of point $\mathbf{p}$ onto plane $\mathbf{g}$. $\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{★}) = g_w^2 (p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3)$ $- (g_x p_x + g_y p_y + g_z p_z)\,g_w \mathbf{e}_4$	
Central projection of point $\mathbf{p}$ onto line $\boldsymbol{l}$. $\boldsymbol{l} \vee (\mathbf{p} \wedge \boldsymbol{l}^{★}) = (l_{mx}^2 + l_{my}^2 + l_{mz}^2)(p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3)$ $- (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)(l_{mx} \mathbf{e}_1 + l_{my} \mathbf{e}_2 + l_{mz} \mathbf{e}_3)$ $+ (l_{mx}(l_{vz} p_y - l_{vy} p_z) + l_{my}(l_{vx} p_z - l_{vz} p_x) + l_{mz}(l_{vy} p_x - l_{vx} p_y))\,\mathbf{e}_4$	
Central projection of line $\boldsymbol{l}$ onto plane $\mathbf{g}$. $\mathbf{g} \vee (\boldsymbol{l} \wedge \mathbf{g}^{★}) = (g_y l_{mz} - g_z l_{my})\,g_w \mathbf{e}_{41} + g_w^2 l_{mx} \mathbf{e}_{23}$ $+ (g_z l_{mx} - g_x l_{mz})\,g_w \mathbf{e}_{42} + g_w^2 l_{my} \mathbf{e}_{31}$ $+ (g_x l_{my} - g_y l_{mx})\,g_w \mathbf{e}_{43} + g_w^2 l_{mz} \mathbf{e}_{12}$	

Motor $\mathbf{Q}$

Motion Operator

$$\mathbf{Q} = \underbrace{Q_{vx} \mathbf{e}_{41} + Q_{vy} \mathbf{e}_{42} + Q_{vz} \mathbf{e}_{43} + Q_{vw} \mathbb{1}}_{\text{Rotation}} + \underbrace{Q_{mx} \mathbf{e}_{23} + Q_{my} \mathbf{e}_{31} + Q_{mz} \mathbf{e}_{12} + Q_{mw} 1}_{\text{Moment and Displacement}}$$

$$\mathbf{Q} = \exp_{⟇}\left[(\delta 1 + \phi \mathbb{1}) ⟇ \boldsymbol{l}\right] = \cos_{⟇}(\delta 1 + \phi \mathbb{1}) + \sin_{⟇}(\delta 1 + \phi \mathbb{1}) ⟇ \boldsymbol{l}$$

$$\mathbf{Q} = \boldsymbol{l} \sin\phi - \boldsymbol{l}^{☆}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

$$\mathbf{Q} = (l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43} + l_{mx} \mathbf{e}_{23} + l_{my} \mathbf{e}_{31} + l_{mz} \mathbf{e}_{12})\sin\phi + (l_{vx} \mathbf{e}_{23} + l_{vy} \mathbf{e}_{31} + l_{vz} \mathbf{e}_{12})\,\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

$\mathbf{Q} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{Q}}$ rotates object $\mathbf{u}$ through angle 2ϕ about line $\boldsymbol{l}$ and translates by distance 2δ along direction of line $\boldsymbol{l}$.

Bulk	$\mathbf{Q}_{\bullet} = Q_{mx} \mathbf{e}_{23} + Q_{my} \mathbf{e}_{31} + Q_{mz} \mathbf{e}_{12} + Q_{mw} 1$
Weight	$\mathbf{Q}_{\circ} = Q_{vx} \mathbf{e}_{41} + Q_{vy} \mathbf{e}_{42} + Q_{vz} \mathbf{e}_{43} + Q_{vw} \mathbb{1}$
Bulk dual	$\mathbf{Q}^{★} = -Q_{mx} \mathbf{e}_{41} - Q_{my} \mathbf{e}_{42} - Q_{mz} \mathbf{e}_{43} + Q_{mw} \mathbb{1}$
Weight dual	$\mathbf{Q}^{☆} = -Q_{vx} \mathbf{e}_{23} - Q_{vy} \mathbf{e}_{31} - Q_{vz} \mathbf{e}_{12} + Q_{vw} 1$
Bulk norm	$\left\Vert\mathbf{Q}\right\Vert_{\bullet} = 1\sqrt{Q_{mx}^2 + Q_{my}^2 + Q_{mz}^2 + Q_{mw}^2}$
Weight norm	$\left\Vert\mathbf{Q}\right\Vert_{\circ} = \mathbb{1}\sqrt{Q_{vx}^2 + Q_{vy}^2 + Q_{vz}^2 + Q_{vw}^2}$

$$\sqrt{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\sqrt{2 + 2Q_{vw}}} ⟇ \left(\mathbb{1} - \frac{Q_{mw}}{2 + 2Q_{vw}}\right)$$

$\mathbf{R} = (a_x \mathbf{e}_{41} + a_y \mathbf{e}_{42} + a_z \mathbf{e}_{43})\sin\phi + \mathbb{1}\cos\phi$	$\mathbf{R} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{R}}$ rotates object $\mathbf{u}$ through angle 2ϕ about unit axis $\mathbf{a}$ through origin.
$\mathbf{S} = \boldsymbol{l}\sin\phi + \mathbb{1}\cos\phi$	$\mathbf{S} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{S}}$ rotates object $\mathbf{u}$ through angle 2ϕ about unitized line $\boldsymbol{l}$.
$\mathbf{T} = \tau_x \mathbf{e}_{23} + \tau_y \mathbf{e}_{31} + \tau_z \mathbf{e}_{12} + \mathbb{1}$	$\mathbf{T} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{T}}$ translates object $\mathbf{u}$ by vector $2\boldsymbol{\tau}$.
Line $\boldsymbol{l}$	$\boldsymbol{l} ⟇ \mathbf{u} ⟇ \underset{\sim}{\boldsymbol{l}}$ rotates object $\mathbf{u}$ through 180° about unitized line $\boldsymbol{l}$.
Matrix conversion	$\mathbf{M}_{\mathbf{Q}} = \mathbf{A}_{\mathbf{Q}} + \mathbf{B}_{\mathbf{Q}}$ $\qquad \mathbf{M}_{\mathbf{Q}}^{-1} = \mathbf{A}_{\mathbf{Q}} - \mathbf{B}_{\mathbf{Q}}$

$$\mathbf{A}_{\mathbf{Q}} = \begin{bmatrix} 1 - 2(Q_{vy}^2 + Q_{vz}^2) & 2Q_{vx}Q_{vy} & 2Q_{vz}Q_{vx} & 2(Q_{vy}Q_{mz} - Q_{vz}Q_{my})\\ 2Q_{vx}Q_{vy} & 1 - 2(Q_{vz}^2 + Q_{vx}^2) & 2Q_{vy}Q_{vz} & 2(Q_{vz}Q_{mx} - Q_{vx}Q_{mz})\\ 2Q_{vz}Q_{vx} & 2Q_{vy}Q_{vz} & 1 - 2(Q_{vx}^2 + Q_{vy}^2) & 2(Q_{vx}Q_{my} - Q_{vy}Q_{mx})\\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B}_{\mathbf{Q}} = \begin{bmatrix} 0 & -2Q_{vz}Q_{vw} & 2Q_{vy}Q_{vw} & 2(Q_{vx}Q_{mw} - Q_{vw}Q_{mx})\\ 2Q_{vz}Q_{vw} & 0 & -2Q_{vx}Q_{vw} & 2(Q_{vy}Q_{mw} - Q_{vw}Q_{my})\\ -2Q_{vy}Q_{vw} & 2Q_{vx}Q_{vw} & 0 & 2(Q_{vz}Q_{mw} - Q_{vw}Q_{mz})\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Flector $\mathbf{F}$

Reflection Operator

$$\mathbf{F} = \underbrace{F_{px} \mathbf{e}_1 + F_{py} \mathbf{e}_2 + F_{pz} \mathbf{e}_3 + F_{pw} \mathbf{e}_4}_{\text{Point}} + \underbrace{F_{gx} \mathbf{e}_{423} + F_{gy} \mathbf{e}_{431} + F_{gz} \mathbf{e}_{412} + F_{gw} \mathbf{e}_{321}}_{\text{Plane}}$$

$$\mathbf{F} = \mathbf{p}\sin\phi + \mathbf{g}\cos\phi$$

$$\mathbf{F} = (p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4)\sin\phi + (g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} + g_w \mathbf{e}_{321})\cos\phi$$

$-\mathbf{F} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{F}}$ rotates object $\mathbf{u}$ through angle 2ϕ about line perpendicular to plane $\mathbf{g}$ passing through point $\mathbf{p}$ and reflects across plane $\mathbf{g}$.

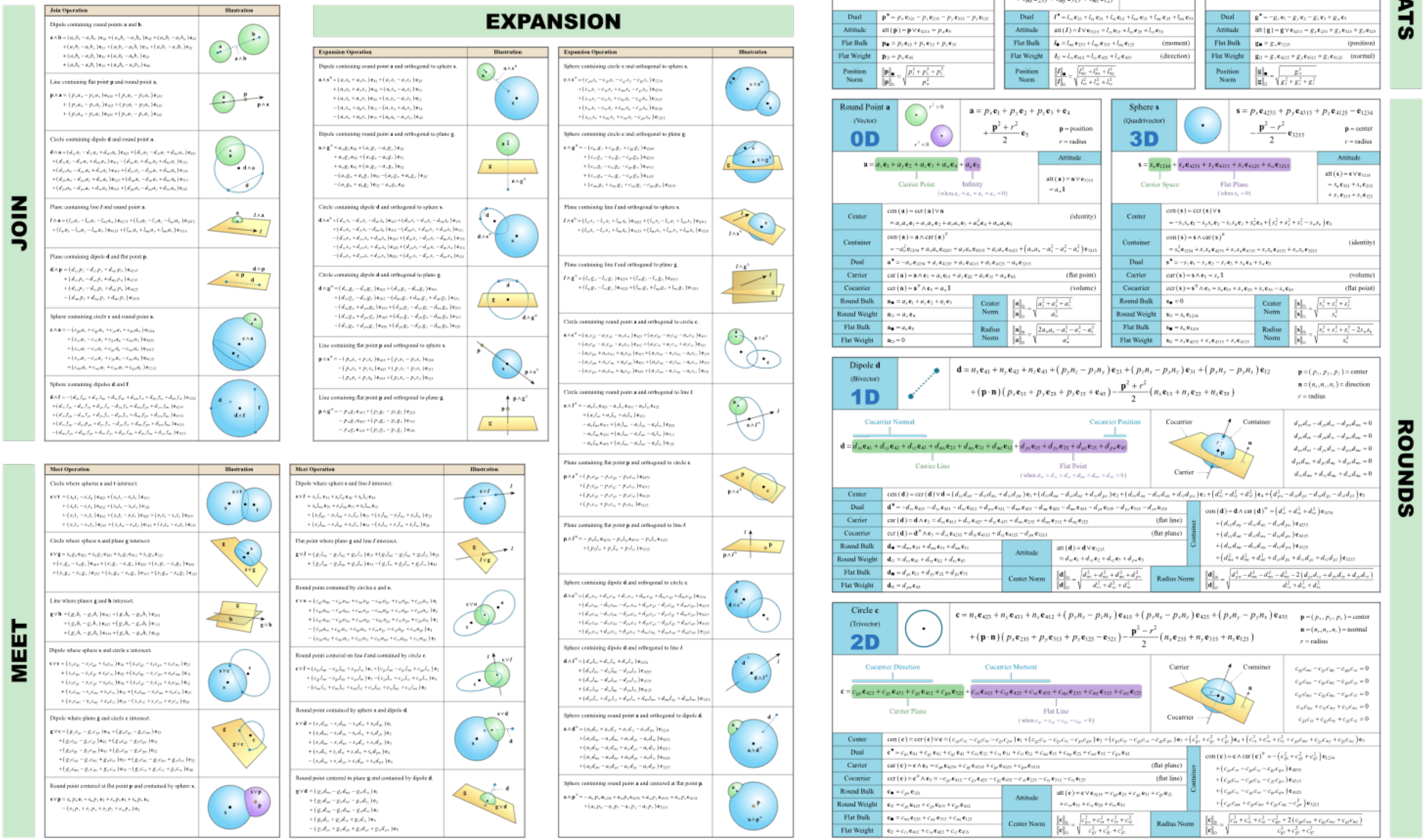
Bulk	$\mathbf{F}_{\bullet} = F_{px} \mathbf{e}_1 + F_{py} \mathbf{e}_2 + F_{pz} \mathbf{e}_3 + F_{gw} \mathbf{e}_{321}$
Weight	$\mathbf{F}_{\circ} = F_{pw} \mathbf{e}_4 + F_{gx} \mathbf{e}_{423} + F_{gy} \mathbf{e}_{431} + F_{gz} \mathbf{e}_{412}$
Bulk dual	$\mathbf{F}^{★} = -F_{gw} \mathbf{e}_4 + F_{px} \mathbf{e}_{423} + F_{py} \mathbf{e}_{431} + F_{pz} \mathbf{e}_{412}$
Weight dual	$\mathbf{F}^{☆} = -F_{gx} \mathbf{e}_1 - F_{gy} \mathbf{e}_2 - F_{gz} \mathbf{e}_3 + F_{pw} \mathbf{e}_{321}$
Bulk norm	$\left\Vert\mathbf{F}\right\Vert_{\bullet} = 1\sqrt{F_{px}^2 + F_{py}^2 + F_{pz}^2 + F_{gw}^2}$
Weight norm	$\left\Vert\mathbf{F}\right\Vert_{\circ} = \mathbb{1}\sqrt{F_{pw}^2 + F_{gx}^2 + F_{gy}^2 + F_{gz}^2}$

$\mathbf{H} = v_x \mathbf{e}_1 + v_y \mathbf{e}_2 + v_z \mathbf{e}_3 + \mathbf{g}$	$-\mathbf{H} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{H}}$ translates object $\mathbf{u}$ by vector $2\mathbf{g}_{xyz} \times \mathbf{v}$ and reflects across unitized plane $\mathbf{g}$ (transflection).
Point $\mathbf{p}$	$-\mathbf{p} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{p}}$ inverts object $\mathbf{u}$ through point $\mathbf{p}$.
Plane $\mathbf{g}$	$-\mathbf{g} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{g}}$ reflects object $\mathbf{u}$ across plane $\mathbf{g}$.
Matrix conversion	$\mathbf{M}_{\mathbf{F}} = \mathbf{A}_{\mathbf{F}} + \mathbf{B}_{\mathbf{F}}$ $\qquad \mathbf{M}_{\mathbf{F}}^{-1} = \mathbf{A}_{\mathbf{F}} - \mathbf{B}_{\mathbf{F}}$

$$\mathbf{A}_{\mathbf{F}} = \begin{bmatrix} 2(F_{gy}^2 + F_{gz}^2) - 1 & -2F_{gx}F_{gy} & -2F_{gz}F_{gx} & 2(F_{gx}F_{gw} - F_{gx}F_{pw})\\ -2F_{gx}F_{gy} & 2(F_{gz}^2 + F_{gx}^2) - 1 & -2F_{gy}F_{gz} & 2(F_{gy}F_{pw} - F_{gy}F_{gw})\\ -2F_{gz}F_{gx} & -2F_{gy}F_{gz} & 2(F_{gx}^2 + F_{gy}^2) - 1 & 2(F_{gz}F_{pw} - F_{gz}F_{gw})\\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B}_{\mathbf{F}} = \begin{bmatrix} 0 & 2F_{gz}F_{pw} & -2F_{gy}F_{pw} & 2(F_{gy}F_{pz} - F_{gz}F_{py})\\ -2F_{gz}F_{pw} & 0 & 2F_{gx}F_{pw} & 2(F_{gz}F_{px} - F_{gx}F_{pz})\\ 2F_{gy}F_{pw} & -2F_{gx}F_{pw} & 0 & 2(F_{gx}F_{py} - F_{gy}F_{px})\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Conformal Geometric Algebra

conformalgeometricalgebra.org



JOIN

MEET

EXPANSION

FLATS

ROUNDS

Flat Point p (Bivector) **0D**

$$\mathbf{p} = p_x\mathbf{e}_{15} + p_y\mathbf{e}_{25} + p_z\mathbf{e}_{35} + p_w\mathbf{e}_{45}$$

Flat Line *l* (Trivector) **1D**

$$\boldsymbol{l} = l_{vx}\mathbf{e}_{415} + l_{vy}\mathbf{e}_{425} + l_{vz}\mathbf{e}_{435} + l_{mx}\mathbf{e}_{235} + l_{my}\mathbf{e}_{315} + l_{mz}\mathbf{e}_{125}$$

Flat Plane g (Quadrivector) **2D**

Round Point a (Vector) **0D**

$$\mathbf{a} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + \mathbf{e}_4 + \frac{\mathbf{p}^2 + r^2}{2}\mathbf{e}_5$$

p = position

r = radius

Sphere s (Quadrivector) **3D**

Dipole d (Bivector) **1D**

Circle c (Trivector) **2D**

A Vast Subject Area

- No hope of covering all the fundamentals in one hour

- This talk is an introduction that paints the big picture

- Tutorial sessions today and Wednesday provide more details
    - Mon, 4:50 – Foundations of Projective Exterior Algebra
    - Wed, 3:20 – Foundations of Projective Geometric Algebra
    - Wed, 4:20 – Applications of Geometric Algebra (Russell Carpenter)

Grassmann / Clifford Algebras

- You’ve probably been using pieces of these algebras already without realizing it

- Cross products
- Homogeneous coordinates (*x*, *y*, *z*, *w*)
- Planes (*a*, *b*, *c*, *d*)
- Plücker coordinates
- Quaternions

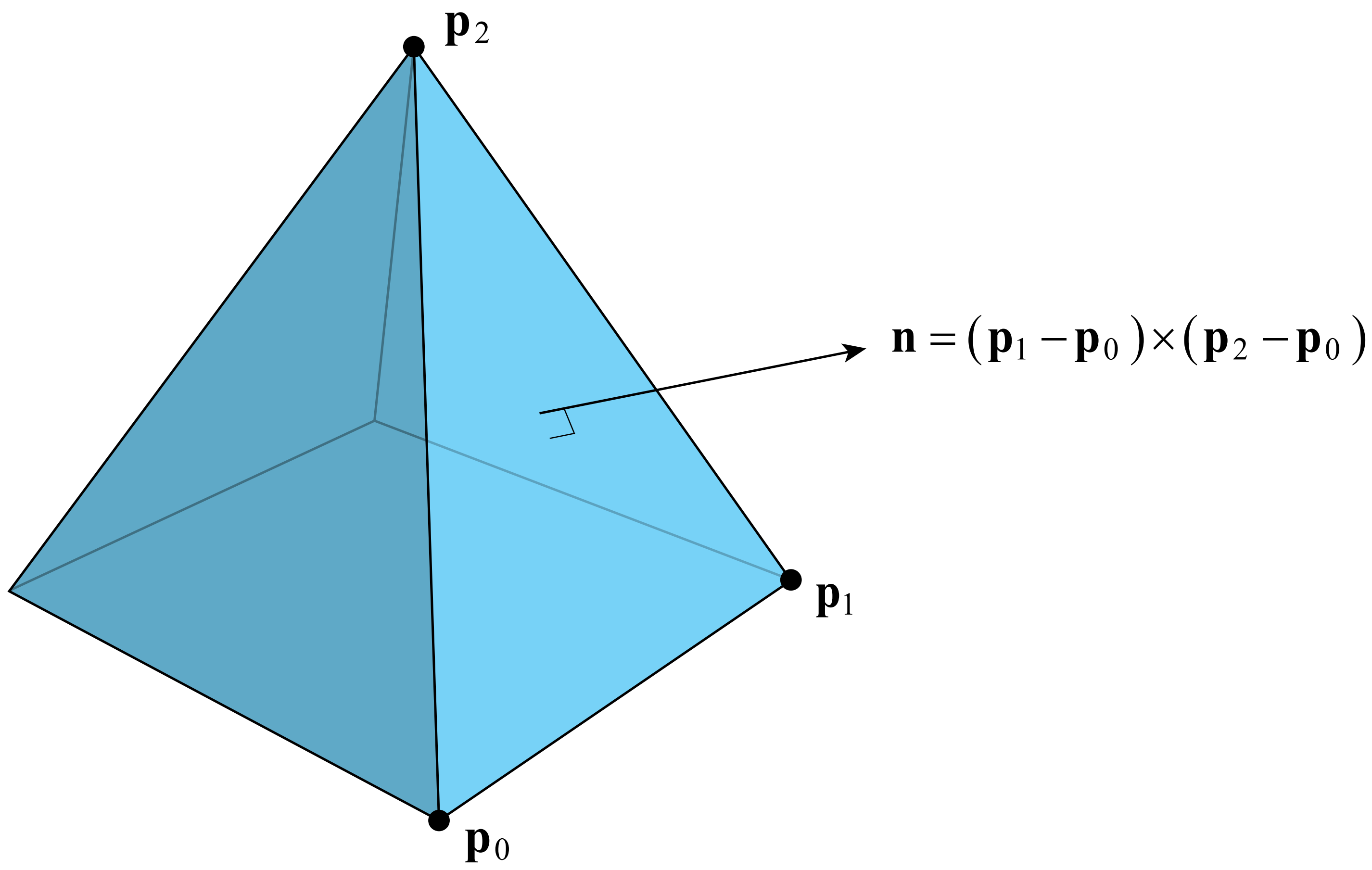
Cross Products

- Units of distance become units of area

$$\begin{tikzcd}
(a_x, a_y, a_z) \times (b_x, b_y, b_z) \arrow[d] \\
(a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x)
\end{tikzcd}$$

Normal Vectors

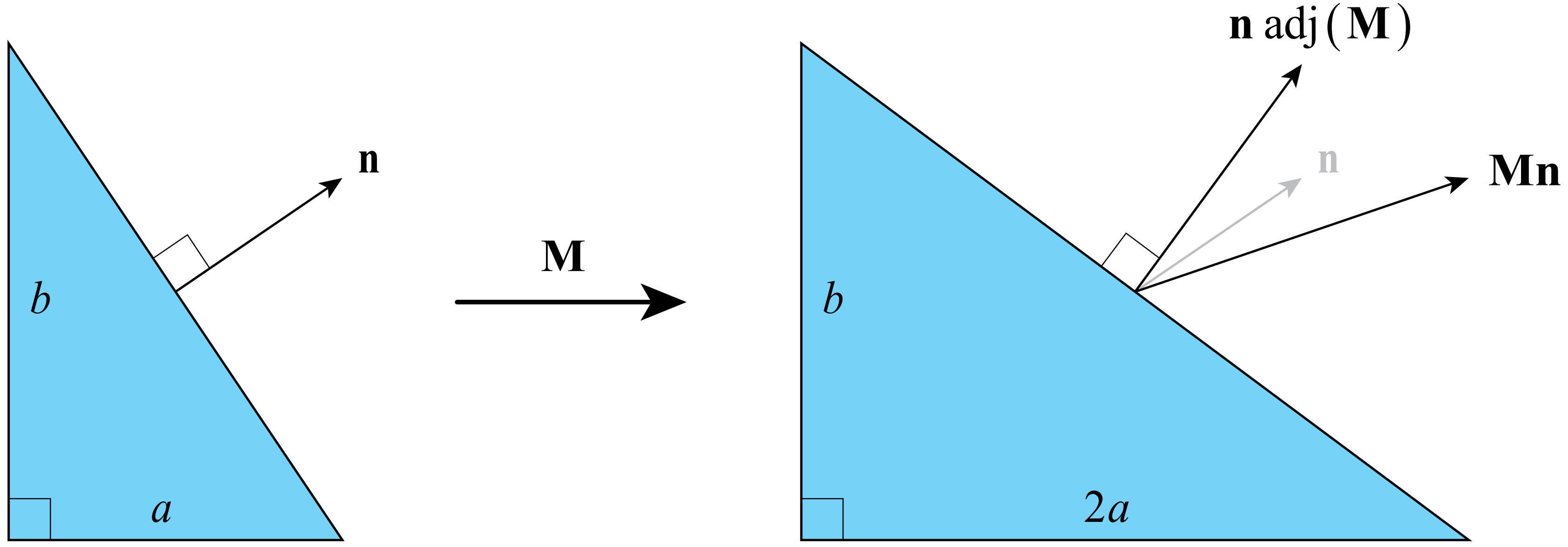
- Cross product calculates normal of triangular face



Normal Vector Transformation

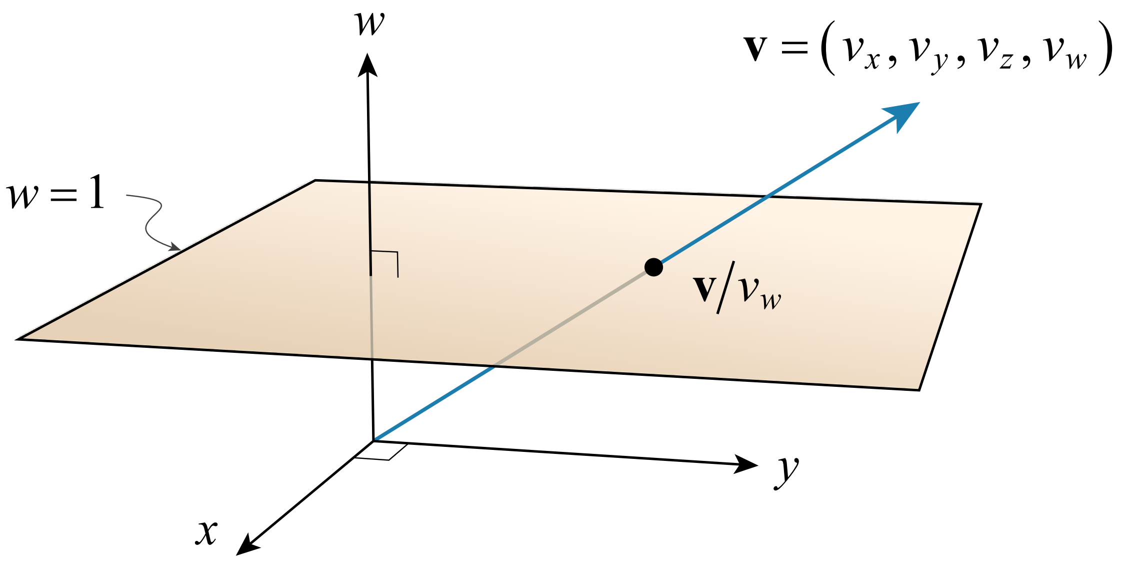
- Normals don’t transform like ordinary vectors
- That’s because they’re something else called *bivectors*

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Homogeneous Coordinates

- 3D points are projections of 4D vectors



Homogeneous Coordinates

- Allows translations to be added to linear transformations

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

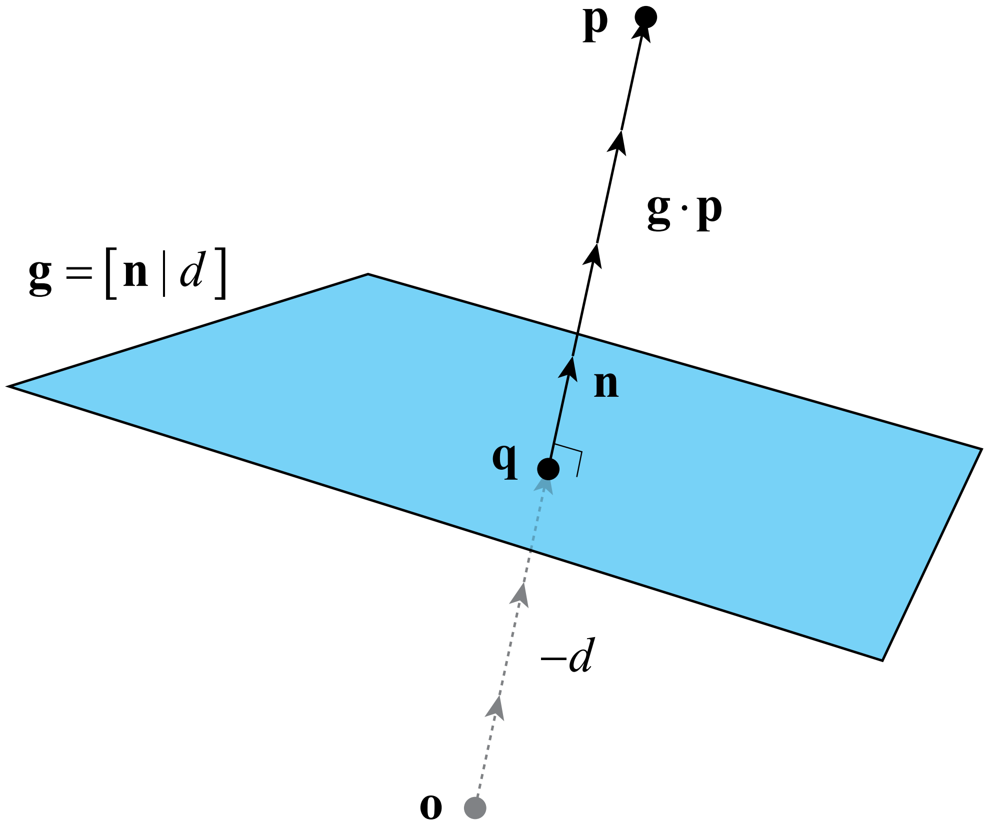
$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & t_x \\ m_{21} & m_{22} & m_{23} & t_y \\ m_{31} & m_{32} & m_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Planes

- 4D dot product with point **p** gives signed distance to plane **g**

$$\mathbf{p} = (x, y, z, w)$$

$$\mathbf{g} = (n_x, n_y, n_z, d)$$



Plücker Coordinates

- Implicit representation of a line in 3D space
- Has 6 coordinates, 3 for direction **v** and 3 for moment **m**

- Given homogeneous points **p** and **q** on the line,

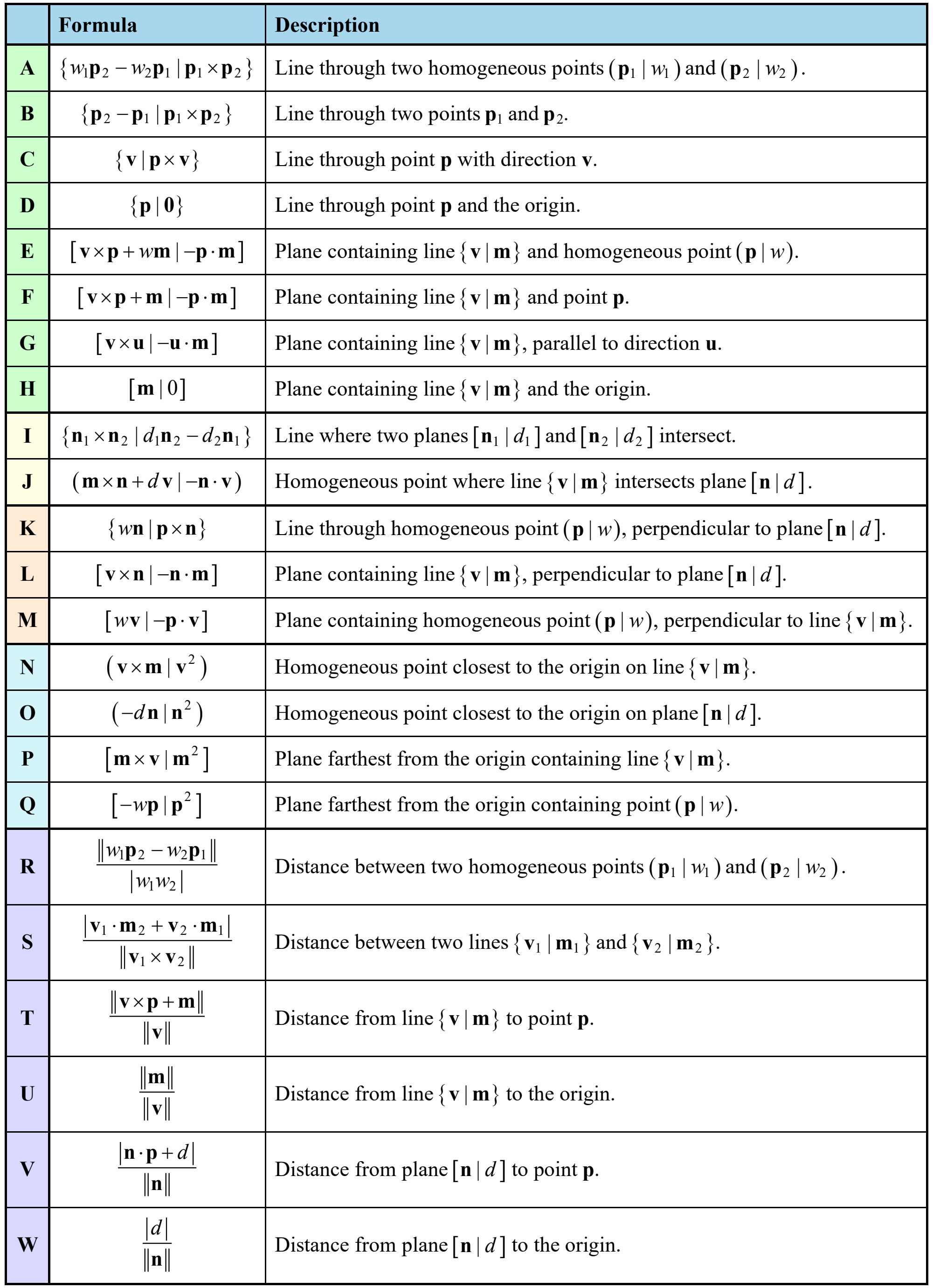
$$\mathbf{v} = p_w\mathbf{q}_{xyz} - q_w\mathbf{p}_{xyz}$$

$$\mathbf{m} = \mathbf{p}_{xyz} \times \mathbf{q}_{xyz}$$

- Same results for any two points spaced same distance apart
- Information about specific points is eliminated

Points, Lines, Planes

- Lots of formulas for combining geometries
- Discovered without knowledge of bigger picture
- We can better explain where all of these formulas come from



	Formula	Description
A	$\{w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two homogeneous points $(\mathbf{p}_1 \mid w_1)$ and $(\mathbf{p}_2 \mid w_2)$.
B	$\{\mathbf{p}_2 - \mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two points $\mathbf{p}_1$ and $\mathbf{p}_2$.
C	$\{\mathbf{v} \mid \mathbf{p} \times \mathbf{v}\}$	Line through point $\mathbf{p}$ with direction $\mathbf{v}$.
D	$\{\mathbf{p} \mid \mathbf{0}\}$	Line through point $\mathbf{p}$ and the origin.
E	$[\mathbf{v} \times \mathbf{p} + w\mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and homogeneous point $(\mathbf{p} \mid w)$.
F	$[\mathbf{v} \times \mathbf{p} + \mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and point $\mathbf{p}$.
G	$[\mathbf{v} \times \mathbf{u} \mid -\mathbf{u} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$, parallel to direction $\mathbf{u}$.
H	$[\mathbf{m} \mid 0]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and the origin.
I	$\{\mathbf{n}_1 \times \mathbf{n}_2 \mid d_1\mathbf{n}_2 - d_2\mathbf{n}_1\}$	Line where two planes $[\mathbf{n}_1 \mid d_1]$ and $[\mathbf{n}_2 \mid d_2]$ intersect.
J	$(\mathbf{m} \times \mathbf{n} + d\,\mathbf{v} \mid -\mathbf{n} \cdot \mathbf{v})$	Homogeneous point where line $\{\mathbf{v} \mid \mathbf{m}\}$ intersects plane $[\mathbf{n} \mid d]$.
K	$\{w\mathbf{n} \mid \mathbf{p} \times \mathbf{n}\}$	Line through homogeneous point $(\mathbf{p} \mid w)$, perpendicular to plane $[\mathbf{n} \mid d]$.
L	$[\mathbf{v} \times \mathbf{n} \mid -\mathbf{n} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$, perpendicular to plane $[\mathbf{n} \mid d]$.
M	$[w\mathbf{v} \mid -\mathbf{p} \cdot \mathbf{v}]$	Plane containing homogeneous point $(\mathbf{p} \mid w)$, perpendicular to line $\{\mathbf{v} \mid \mathbf{m}\}$.
N	$(\mathbf{v} \times \mathbf{m} \mid \mathbf{v}^2)$	Homogeneous point closest to the origin on line $\{\mathbf{v} \mid \mathbf{m}\}$.
O	$(-d\mathbf{n} \mid \mathbf{n}^2)$	Homogeneous point closest to the origin on plane $[\mathbf{n} \mid d]$.
P	$[\mathbf{m} \times \mathbf{v} \mid \mathbf{m}^2]$	Plane farthest from the origin containing line $\{\mathbf{v} \mid \mathbf{m}\}$.
Q	$[-w\mathbf{p} \mid \mathbf{p}^2]$	Plane farthest from the origin containing point $(\mathbf{p} \mid w)$.
R	$\dfrac{\lVert w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \rVert}{\lvert w_1 w_2 \rvert}$	Distance between two homogeneous points $(\mathbf{p}_1 \mid w_1)$ and $(\mathbf{p}_2 \mid w_2)$.
S	$\dfrac{\lvert \mathbf{v}_1 \cdot \mathbf{m}_2 + \mathbf{v}_2 \cdot \mathbf{m}_1 \rvert}{\lVert \mathbf{v}_1 \times \mathbf{v}_2 \rVert}$	Distance between two lines $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ and $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$.
T	$\dfrac{\lVert \mathbf{v} \times \mathbf{p} + \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to point $\mathbf{p}$.
U	$\dfrac{\lVert \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to the origin.
V	$\dfrac{\lvert \mathbf{n} \cdot \mathbf{p} + d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to point $\mathbf{p}$.
W	$\dfrac{\lvert d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to the origin.

Quaternions

- A quaternion **q** represents a rotation in 3D space

$$\mathbf{q} = xi + yj + zk + w \qquad i^2 = j^2 = k^2 = -1 \qquad \begin{aligned} ij &= -ji = k \\ jk &= -kj = i \\ ki &= -ik = j \end{aligned}$$

- Rotation through angle ϕ about axis **a** is

$$\mathbf{q} = \left(\sin\frac{\phi}{2} \right) \mathbf{a} + \cos\frac{\phi}{2}$$

Quaternions

- A quaternion rotates a vector **v** with the sandwich product

$$\mathbf{v}' = \mathbf{q}\mathbf{v}\mathbf{q}^{*} \qquad \mathbf{v} = v_x i + v_y j + v_z k$$

- **q*** is the conjugate of the quaternion:

$$\mathbf{q} = -xi - yj - zk + w$$

All Part of Same Algebraic Structure

- Non-vector result of cross product
- 4D homogeneous coordinates for points
- 6D Plücker coordinates for lines
- 4D plane representations
- Quaternions

4D Projective Algebras

- 4D rigid exterior algebra
  - Homogeneous representation of 3D geometry
  - Points, lines, planes
  - Join, meet, projection, norm, distance, angle

- 4D rigid geometric algebra
  - Euclidean isometries in 3D space
  - Rotations, translations, screw transformations
  - Parameterization, interpolation

Exterior / Grassmann Algebra

- Wedge product ∧
  - Combines dimensions of operands
  - Vectors square to zero:

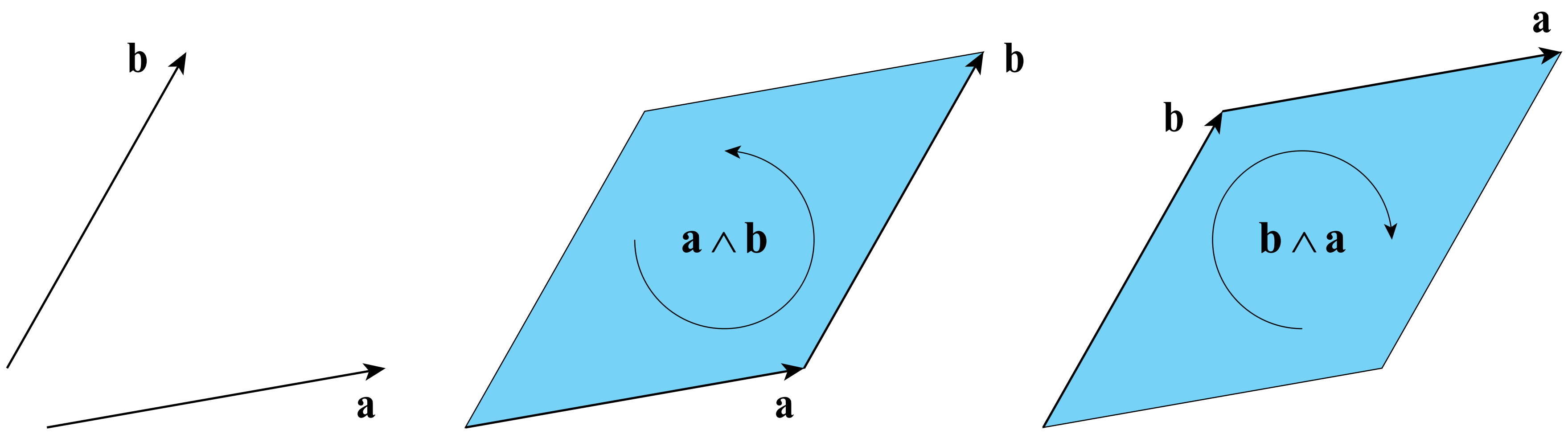
$$\mathbf{v} \wedge \mathbf{v} = 0$$

  - Antisymmetric on vectors:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

Bivectors

- Wedge product of two vectors **a** and **b**



Bivectors

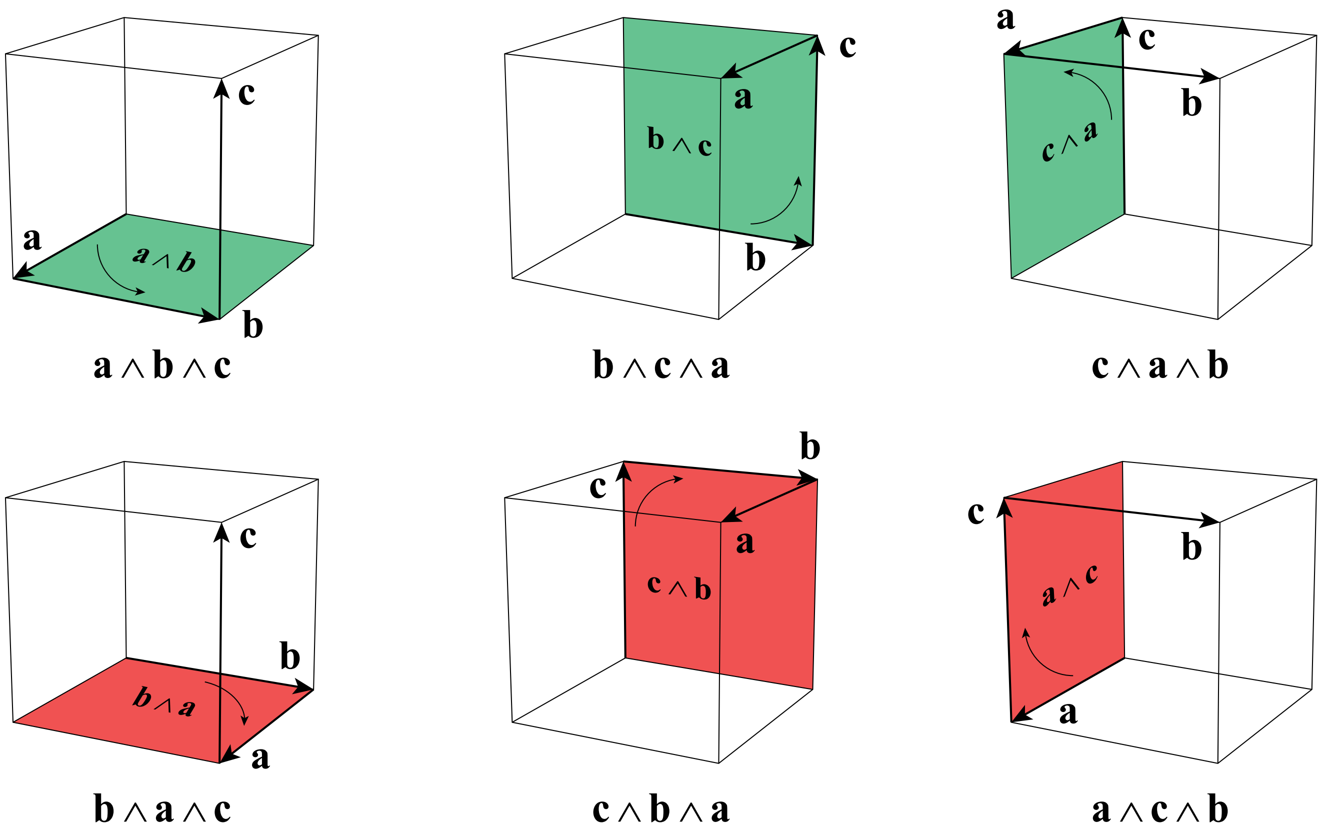
- Wedge product of two vectors **a** and **b**:

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b} = &\left(a_y b_z - a_z b_y\right)\left(\mathbf{e}_2\wedge\mathbf{e}_3\right)\\ &+\left(a_z b_x - a_x b_z\right)\left(\mathbf{e}_3\wedge\mathbf{e}_1\right)\\ &+\left(a_x b_y - a_y b_x\right)\left(\mathbf{e}_1\wedge\mathbf{e}_2\right)\end{aligned}$$

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b} = &\left(a_y b_z - a_z b_y\right)\mathbf{e}_{23}\\ &+\left(a_z b_x - a_x b_z\right)\mathbf{e}_{31}\\ &+\left(a_x b_y - a_y b_x\right)\mathbf{e}_{12}\end{aligned}$$

Trivectors

- Wedge product of three vectors **a**, **b**, and **c**



Trivectors

- Wedge product of three vectors **a**, **b**, and **c**

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \left(a_x b_y c_z + a_y b_z c_x + a_z b_x c_y - a_x b_z c_y - a_y b_x c_z - a_z b_y c_x \right) \mathbf{e}_{123}$$

- Determinant of 3 × 3 matrix with columns **a**, **b**, and **c**

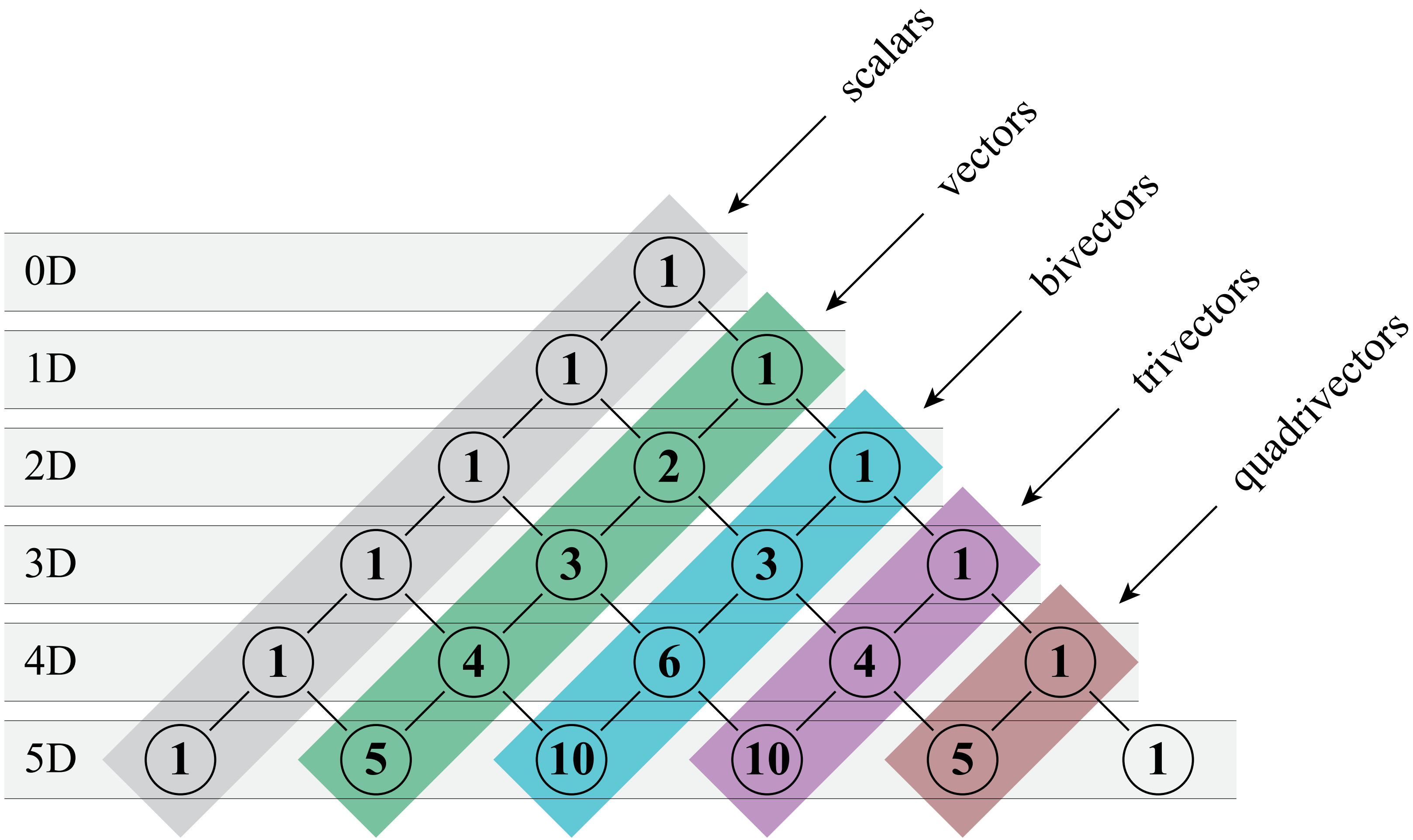
3D Vector Space

Scalars	s	Magnitudes
Vectors	$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$	Directed lengths

3D Exterior Algebra

Scalars	s	Magnitudes
Vectors	$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$	Directed lengths
Bivectors	$x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12}$	Directed areas
Trivectors	t	Directed volumes

Pascal's Triangle



Rigid Exterior / Geometric Algebra

- Projective algebra with one extra dimension
- Contains points, lines, planes in 3D
- Can perform rotations, translations, screw transformations

4D Exterior Algebra

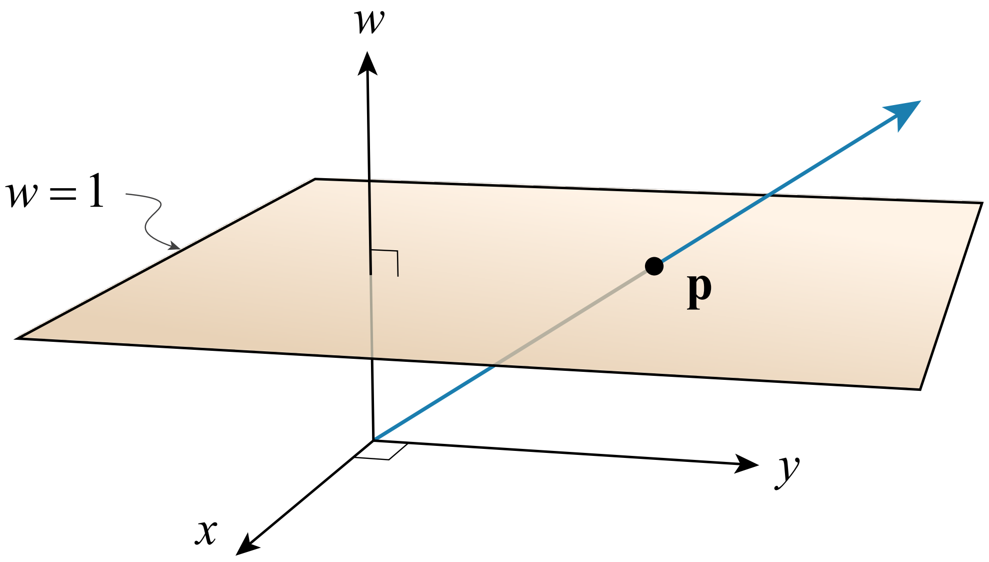
- Extends 4D vector space

- One scalar **1**
- Four vector basis elements
- Six bivector basis elements
- Four trivector basis elements
- One antiscalar $\mathbb{1}$

Type	Values	Grade / Antigrade	
Scalar	$\mathbf{1}$	0 / 4	▯▯▯▯
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4 = \mathbf{e}_n$	1 / 3	▮▯▯▯ ▯▮▯▯ ▯▯▮▯ ▯▯▯▮
Bivectors	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$	2 / 2	▮▯▯▮ ▯▮▯▮ ▯▯▮▮ ▯▮▮▯ ▮▯▮▯ ▮▮▯▯
Trivectors / Antivectors	$\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$	3 / 1	▯▮▮▮ ▮▯▮▮ ▮▮▯▮ ▮▮▮▯
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	▮▮▮▮

Point

$$\mathbf{p} = \underbrace{p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3}_{\text{Position}} + \underbrace{p_w\mathbf{e}_4}_{\text{Weight}}$$



Special Points

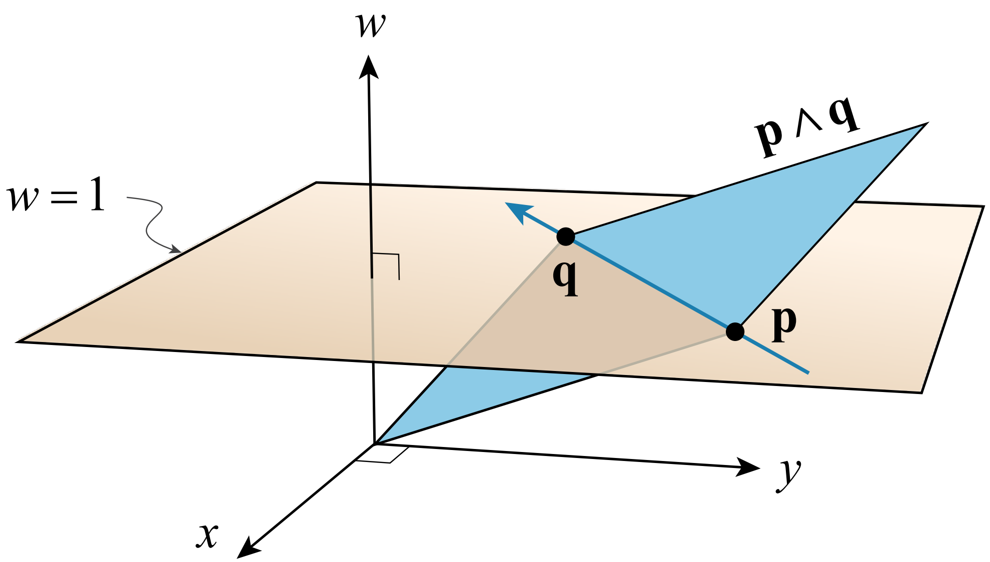
- The origin is simply the point $\mathbf{e}_4$
- Point with zero weight lies at infinity in (x, y, z) direction
- Points at infinity in opposite directions are equivalent

Line

$$\mathbf{p} \wedge \mathbf{q} = (q_x p_w - p_x q_w)\,\mathbf{e}_{41} + (q_y p_w - p_y q_w)\,\mathbf{e}_{42} + (q_z p_w - p_z q_w)\,\mathbf{e}_{43}$$
$$+ (p_y q_z - p_z q_y)\,\mathbf{e}_{23} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}$$

$$\boldsymbol{l} = \underbrace{l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}}_{\text{Moment}}$$

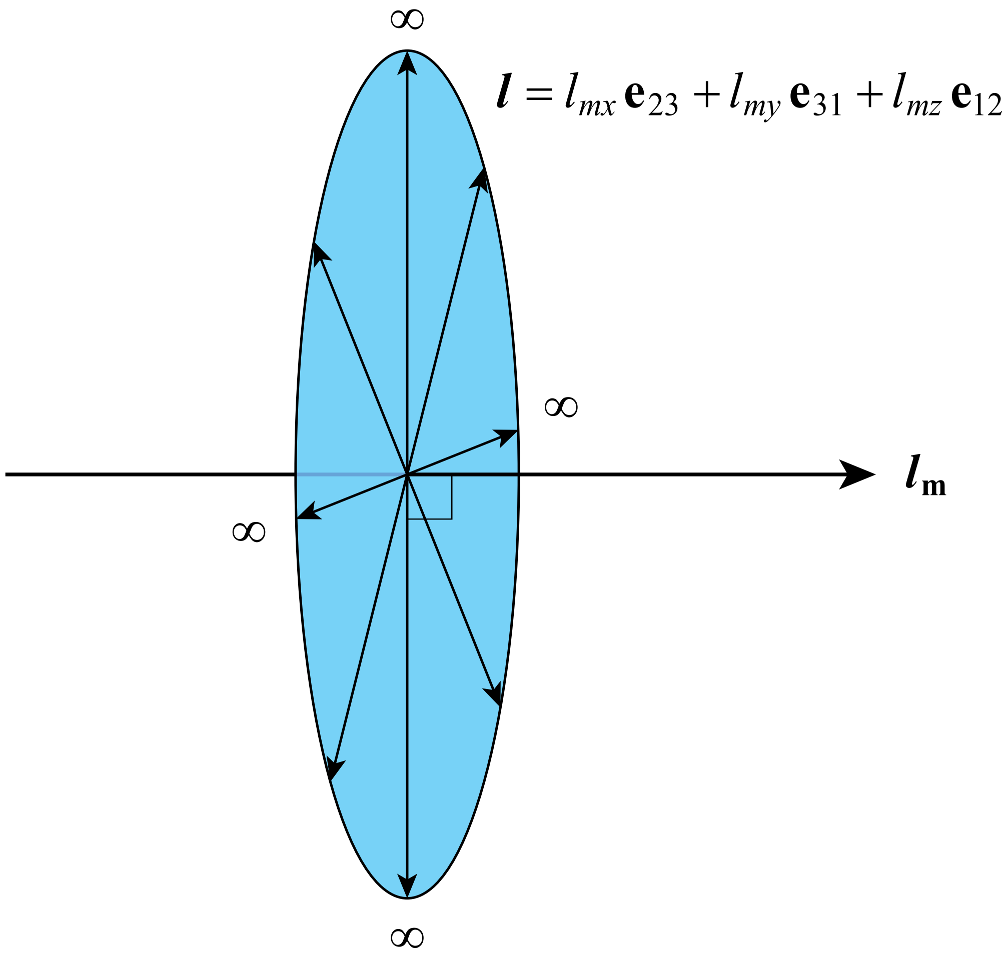
$$\boldsymbol{l}_{\mathbf{v}} \cdot \boldsymbol{l}_{\mathbf{m}} = 0$$



Sec. 2.4.2

Lines at Infinity

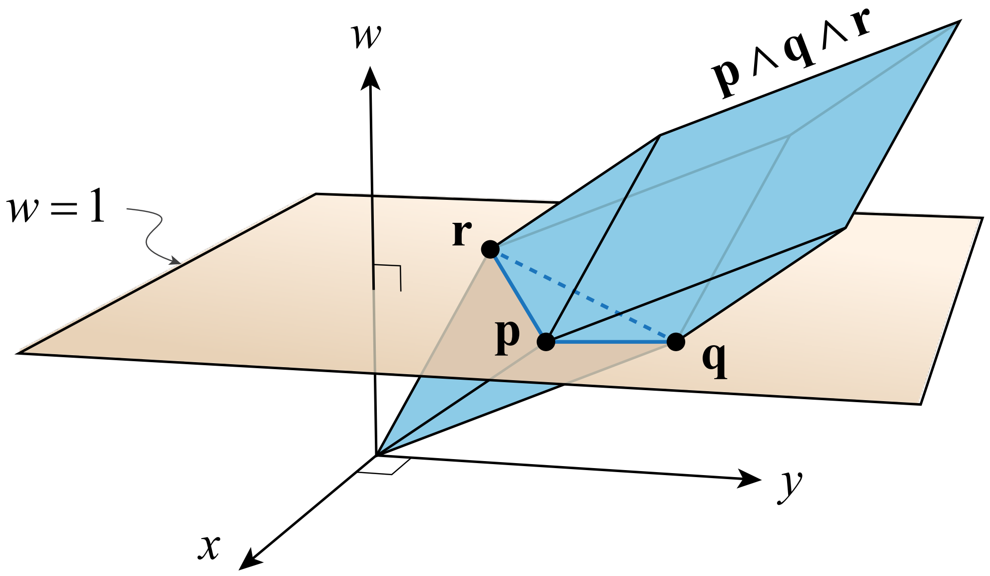
- Line with zero direction lies at infinity



Plane

$$\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx})\,\mathbf{e}_{423} + (l_{vz} p_x - l_{vx} p_z + l_{my})\,\mathbf{e}_{431}$$
$$+ (l_{vx} p_y - l_{vy} p_x + l_{mz})\,\mathbf{e}_{423} - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\,\mathbf{e}_{321}$$

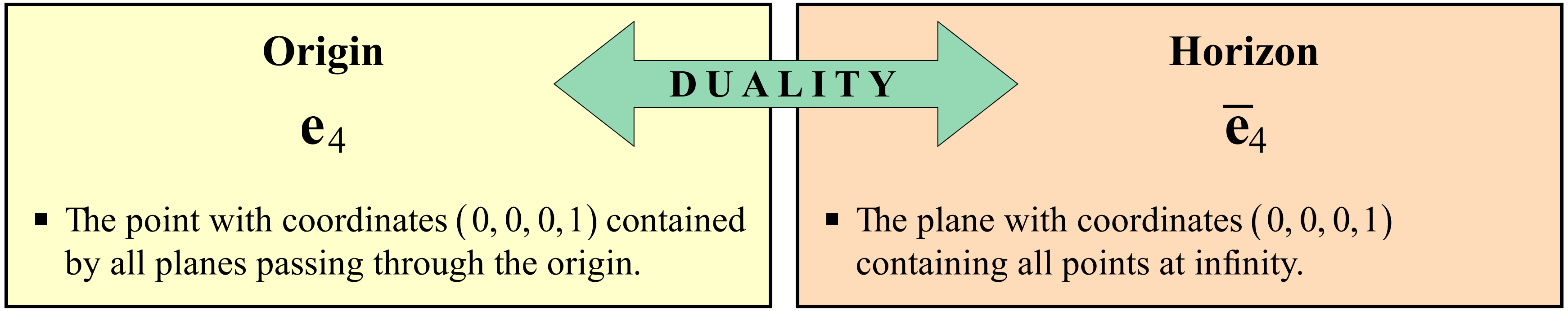
$$\mathbf{g} = \underbrace{g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w\mathbf{e}_{321}}_{\text{Position}}$$



Sec. 2.4.3

Horizon

- Plane with zero normal lies at infinity: $g_w \mathbf{e}_{321}$
- Contains all points at infinity, all lines at infinity
- Given special name *horizon*



Origin

$\mathbf{e}_4$

- The point with coordinates $(0, 0, 0, 1)$ contained by all planes passing through the origin.

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Horizon

$\overline{\mathbf{e}}_4$

- The plane with coordinates $(0, 0, 0, 1)$ containing all points at infinity.

4D Exterior Algebra

Scalars	s	Magnitudes
Vectors	$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_4$	Points
Bivectors	$v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	Lines
Trivectors	$g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321}$	Planes
Quadrivectors	t	Magnitudes

Complements

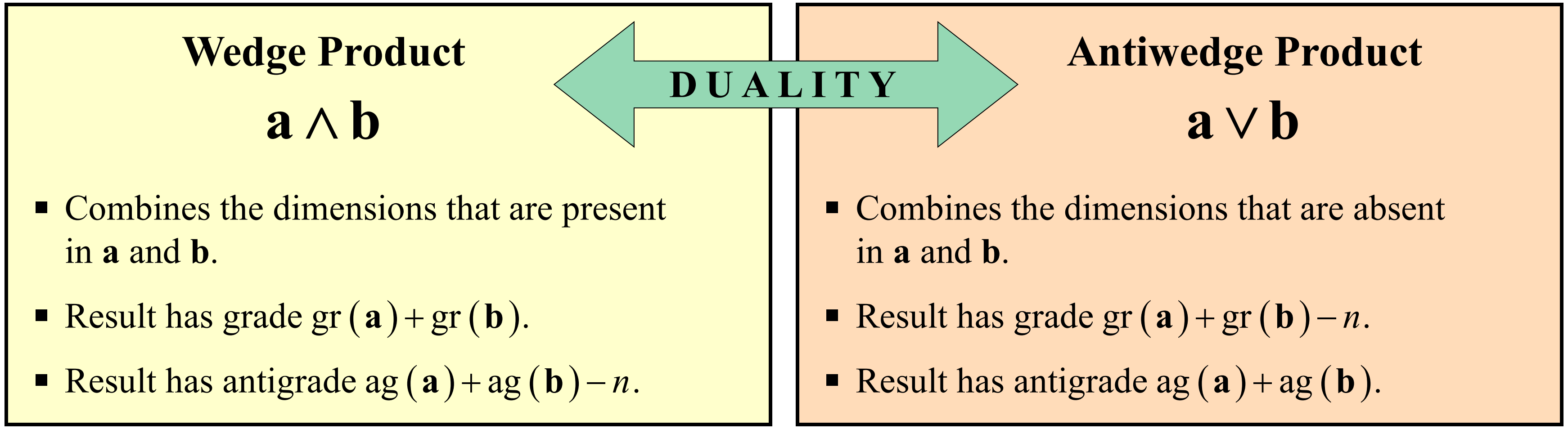
- Complement inverts full / empty dimensions
- Right complement denoted by overbar
- Left complement denoted by underbar
- For basis element **u**,

$$\mathbf{u} \wedge \overline{\mathbf{u}} = \mathbb{1} \qquad \underline{\mathbf{u}} \wedge \mathbf{u} = \mathbb{1}$$

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\overline{\mathbf{u}}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$\mathbf{1}$
$\underline{\mathbf{u}}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{1}$

Antiwedge Product

- Antiwedge product denoted by ∨



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Wedge Product	**Antiwedge Product**
$\mathbf{a} \wedge \mathbf{b}$	$\mathbf{a} \vee \mathbf{b}$
▪ Combines the dimensions that are present in **a** and **b**.	▪ Combines the dimensions that are absent in **a** and **b**.
▪ Result has grade $\operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b})$.	▪ Result has grade $\operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b}) - n$.
▪ Result has antigrade $\operatorname{ag}(\mathbf{a}) + \operatorname{ag}(\mathbf{b}) - n$.	▪ Result has antigrade $\operatorname{ag}(\mathbf{a}) + \operatorname{ag}(\mathbf{b})$.

De Morgan Laws

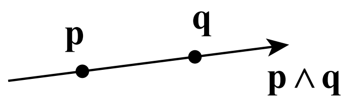
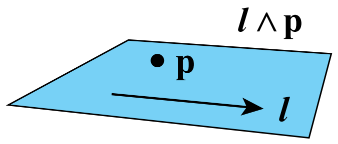
- Every operation with ‘anti’ in its name satisfies a De Morgan law:

$$\overline{\mathbf{a} \vee \mathbf{b}} = \overline{\mathbf{a}} \wedge \overline{\mathbf{b}} \qquad\qquad \underline{\mathbf{a} \vee \mathbf{b}} = \underline{\mathbf{a}} \wedge \underline{\mathbf{b}}$$

- To calculate anti-operation,
  - Take a complement of each input
  - Perform the regular operation
  - Take opposite complement of the result

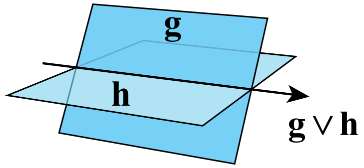
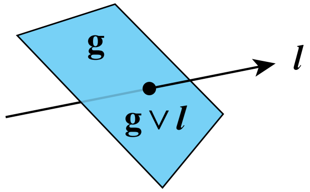
Join

- Wedge product performs join operation
- Produces higher-dimensional object containing both operands

Join Operation	**Illustration**
Line containing points **p** and **q**. $\mathbf{p} \wedge \mathbf{q} = (p_w q_x - p_x q_w)\,\mathbf{e}_{41} + (p_w q_y - p_y q_w)\,\mathbf{e}_{42} + (p_w q_z - p_z q_w)\,\mathbf{e}_{43} + (p_y q_z - p_z q_y)\,\mathbf{e}_{23} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}$	
Plane containing line ***l*** and point **p**. $\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx} p_w)\,\mathbf{e}_{423} + (l_{vz} p_x - l_{vx} p_z + l_{my} p_w)\,\mathbf{e}_{431} + (l_{vx} p_y - l_{vy} p_x + l_{mz} p_w)\,\mathbf{e}_{412} - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\,\mathbf{e}_{321}$	

Meet

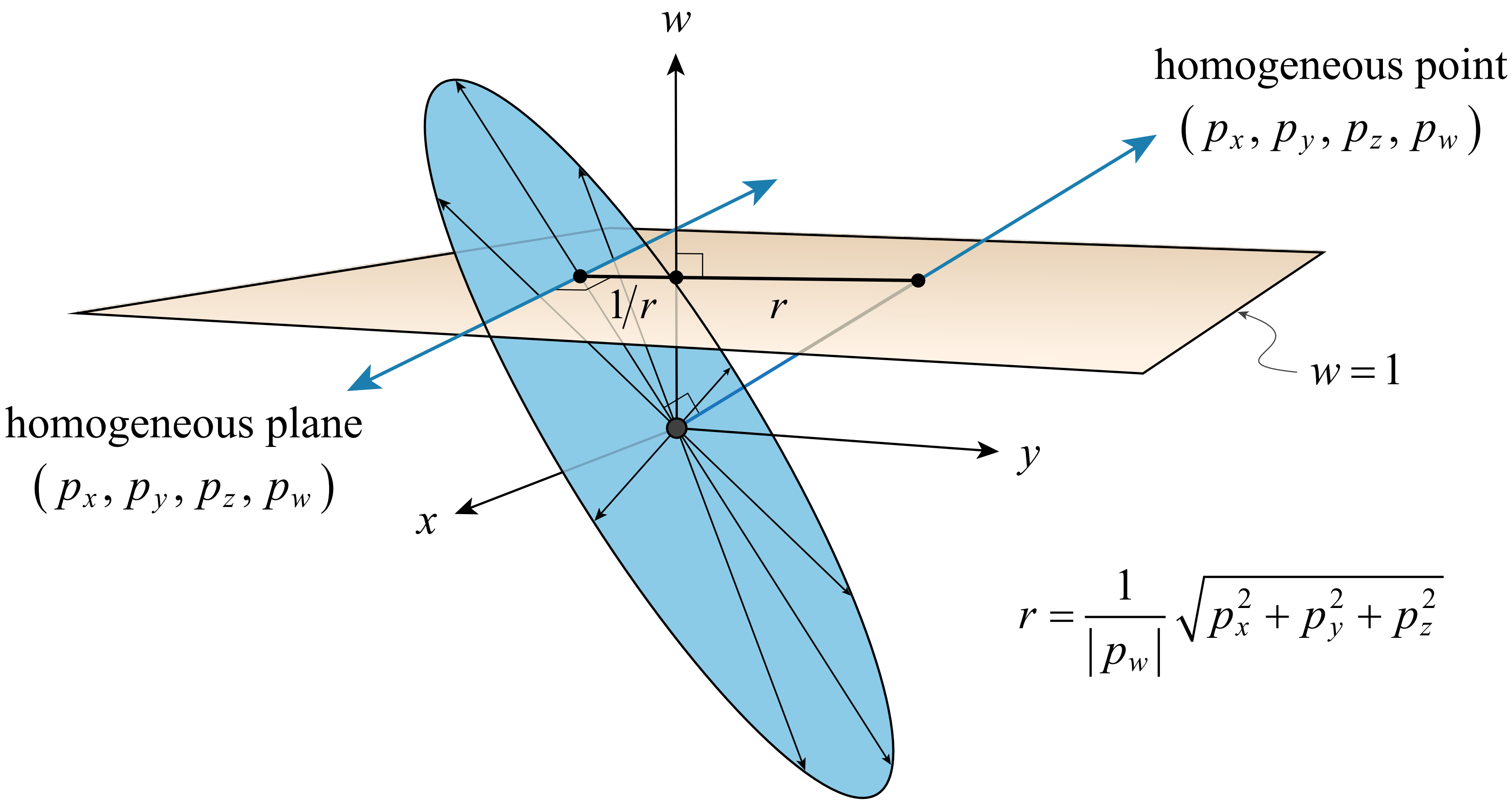
- Antiwedge product performs meet operation
- Produces lower-dimensional object at intersection of operands

Meet Operation	**Illustration**
Line where planes **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\,\mathbf{e}_{41} + (g_x h_z - g_z h_x)\,\mathbf{e}_{42} + (g_y h_x - g_x h_y)\,\mathbf{e}_{43}$ $+ (g_x h_w - g_w h_x)\,\mathbf{e}_{23} + (g_y h_w - g_w h_y)\,\mathbf{e}_{31} + (g_z h_w - g_w h_z)\,\mathbf{e}_{12}$	
Point where plane **g** and line ***l*** intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\,\mathbf{e}_1 + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\,\mathbf{e}_2$ $+ (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\,\mathbf{e}_3 - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\,\mathbf{e}_4$	

Duality

- Every object can be interpreted as two different things
- Every operation performs two different actions
- One interpretation corresponds to regular space
- The other interpretation corresponds to *antispace*

Duality



$$r = \frac{1}{|p_w|}\sqrt{p_x^2 + p_y^2 + p_z^2}$$

The Metric Tensor

- *n* x *n* matrix that defines dot products of vectors

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$

$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$

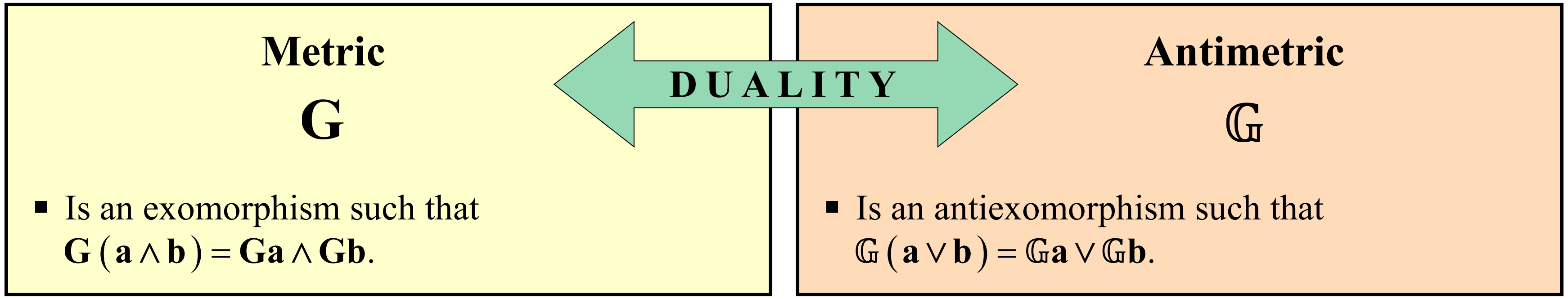
$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_4 \cdot \mathbf{e}_4 = 0$$

$$\mathfrak{g}_{ij} \equiv \mathbf{v}_i \cdot \mathbf{v}_j$$

Metric Exomorphism

- The metric tensor is a linear transformation
- It can be extended to a $2^n \times 2^n$ matrix **G** that applies to entire exterior algebra
- There is also an *antimetric* that satisfies $\mathbb{G}\mathbf{u} = \underline{\mathbf{G}\bar{\mathbf{u}}} = \overline{\mathbf{G}\underline{\mathbf{u}}}$



Metric

$\mathbf{G}$

- Is an exomorphism such that $\mathbf{G}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{Ga} \wedge \mathbf{Gb}$.

DUALITY

Antimetric

$\mathbb{G}$

- Is an antiexomorphism such that $\mathbb{G}(\mathbf{a} \vee \mathbf{b}) = \mathbb{G}\mathbf{a} \vee \mathbb{G}\mathbf{b}$.

Metric and Antimetric

$$
\mathbf{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\hline
\square & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\hline
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square \\
\hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
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\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & \square \\
\hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0
\end{array}\right]
$$

$$
\mathbb{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\hline
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\hline
\square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
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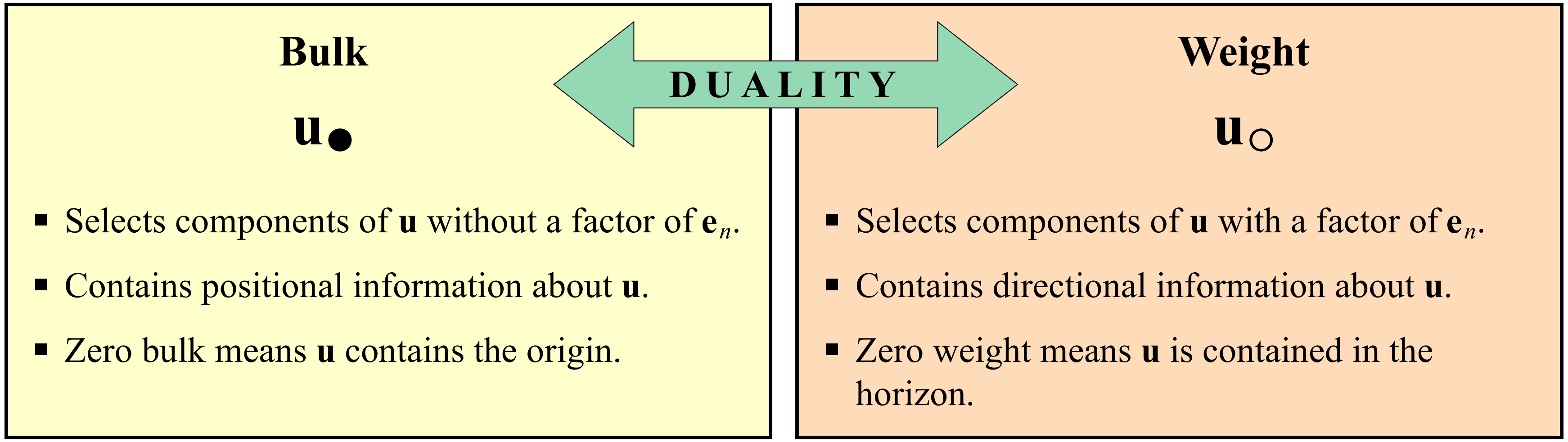
\hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1
\end{array}\right]
$$

$$\mathbf{G}\mathbb{G} = \det(\mathfrak{g})\,\mathbf{I}$$

Bulk and Weight

- Bulk $\mathbf{u}_\bullet = \mathbf{G}\mathbf{u}$ All components without factor $\mathbf{e}_4$
- Weight $\mathbf{u}_\circ = \mathbb{G}\mathbf{u}$ All components with factor $\mathbf{e}_4$



Bulk

$\mathbf{u}_\bullet$

- Selects components of **u** without a factor of $\mathbf{e}_n$.
- Contains positional information about **u**.
- Zero bulk means **u** contains the origin.

Weight

$\mathbf{u}_\circ$

- Selects components of **u** with a factor of $\mathbf{e}_n$.
- Contains directional information about **u**.
- Zero weight means **u** is contained in the horizon.

Sec. 2.8.3

Inner Products

- Dot product defined by metric: $\mathbf{a} \bullet \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}} \mathbf{G} \mathbf{b}\right) \mathbf{1}$
- Antidot product defined by antimetric: $\mathbf{a} \circ \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b}\right) \mathbb{1}$
- Satisfies De Morgan law: $\mathbf{a} \circ \mathbf{b} = \overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}}$

Dot Product

$\mathbf{a} \bullet \mathbf{b}$

- Defined as $\left(\mathbf{a}^{\mathrm{T}} \mathbf{G} \mathbf{b}\right) \mathbf{1}$.
- Inner product between bulks.

DUALITY

Antidot Product

$\mathbf{a} \circ \mathbf{b}$

- Defined as $\left(\mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b}\right) \mathbb{1}$.
- Inner product between weights.

Bulk and Weight Norms

- Two dot products induce two norms
- Bulk norm: $\left\|\mathbf{u}\right\|_{\bullet} = \sqrt{\mathbf{u} \bullet \mathbf{u}}$
- Weight norm: $\left\|\mathbf{u}\right\|_{\circ} = \sqrt{\mathbf{u} \circ \mathbf{u}}$

Bulk and Weight Norms

Type	**Bulk Norm**	**Weight Norm**
Point **p**	$\lVert\mathbf{p}\rVert_\bullet = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2}$	$\lVert\mathbf{p}\rVert_\circ = \lvert p_w \rvert \mathbb{1}$
Line ***l***	$\lVert\boldsymbol{l}\rVert_\bullet = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$	$\lVert\boldsymbol{l}\rVert_\circ = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$
Plane **g**	$\lVert\mathbf{g}\rVert_\bullet = \lvert g_w \rvert \mathbf{1}$	$\lVert\mathbf{g}\rVert_\circ = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$

Unitization

- An object is *unitized* when its weight has magnitude one

Type	**Definition**	**Unitization**
Point $\mathbf{p}$	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	$p_w^2 = 1$
Line $\boldsymbol{l}$	$\boldsymbol{l} = l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43} + l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}$	$l_{vx}^2 + l_{vy}^2 + l_{vz}^2 = 1$
Plane $\mathbf{g}$	$\mathbf{g} = g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321}$	$g_x^2 + g_y^2 + g_z^2 = 1$

Geometric Norm

- Bulk and weight norms by themselves not very meaningful
- But add them, and result is a *homogeneous magnitude*
- Represents distance from origin
- Called the *geometric norm*

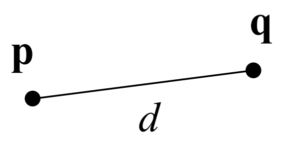
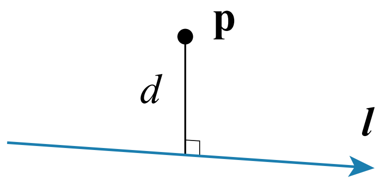
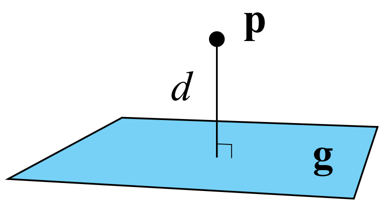
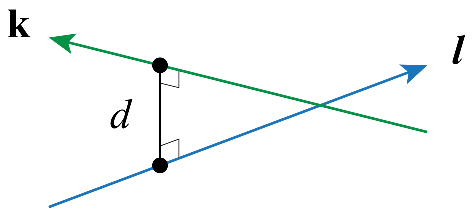
$$\left\|\mathbf{u}\right\| = \left\|\mathbf{u}\right\|_{\bullet} + \left\|\mathbf{u}\right\|_{\circ} = \sqrt{\mathbf{u} \bullet \mathbf{u}} + \sqrt{\mathbf{u} \circ \mathbf{u}}$$

- Two-component quantity, sum of scalar and antiscalar
- Can be unitized by making weight one

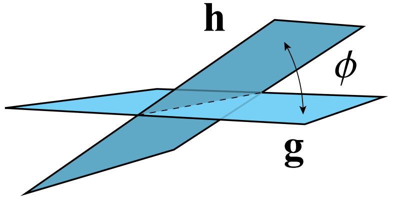
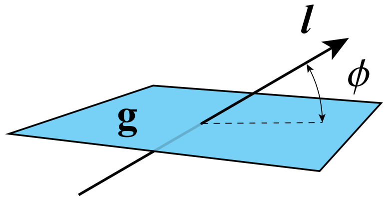
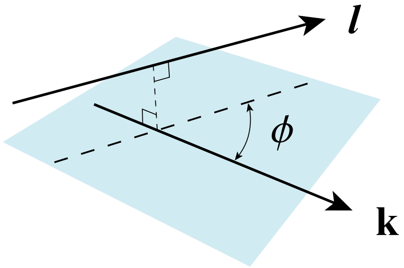
Geometric Norm

Type	Geometric Norm	Interpretation
Point **p**	$\widehat{\lVert\mathbf{p}\rVert} = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from the origin to the point **p**.
Line ***l***	$\widehat{\lVert\boldsymbol{l}\rVert} = \dfrac{\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}}{\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$	Perpendicular distance from the origin to the line ***l***.
Plane **g**	$\widehat{\lVert\mathbf{g}\rVert} = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$	Perpendicular distance from the origin to the plane **g**.

Euclidean Distance

Distance Formula	**Illustration**
Distance *d* between points **p** and **q**. $d(\mathbf{p},\mathbf{q}) = \lVert \mathbf{q}_{xyz} p_w - \mathbf{p}_{xyz} q_w \rVert \mathbf{1} + \lvert p_w q_w \rvert \mathbb{1}$	
Perpendicular distance *d* between point **p** and line ***l***. $d(\mathbf{p},\boldsymbol{l}) = \lVert \boldsymbol{l}_\mathbf{v} \times \mathbf{p}_{xyz} + p_w \boldsymbol{l}_\mathbf{m} \rVert \mathbf{1} + \lVert p_w \boldsymbol{l}_\mathbf{v} \rVert \mathbb{1}$	
Perpendicular distance *d* between point **p** and plane **g**. $d(\mathbf{p},\mathbf{g}) = (\mathbf{p} \cdot \mathbf{g}) \mathbf{1} + \lVert p_w \mathbf{g}_{xyz} \rVert \mathbb{1}$	
Perpendicular distance *d* between skew lines ***l*** and **k**. $d(\boldsymbol{l},\mathbf{k}) = -(\boldsymbol{l}_\mathbf{v} \cdot \mathbf{k}_\mathbf{m} + \boldsymbol{l}_\mathbf{m} \cdot \mathbf{k}_\mathbf{v}) \mathbf{1} + \lVert \boldsymbol{l}_\mathbf{v} \times \mathbf{k}_\mathbf{v} \rVert \mathbb{1}$	

Euclidean Angle

Angle Formula	**Illustration**
Cosine of angle ϕ between planes **g** and **h**. $\cos\phi\,(\mathbf{g},\mathbf{h}) = (\mathbf{g}_{xyz}\cdot\mathbf{h}_{xyz})\,\mathbf{1}+\lVert\mathbf{g}\rVert_{\circ}\lVert\mathbf{h}\rVert_{\circ}$	
Cosine of angle ϕ between plane **g** and line ***l***. $\cos\phi\,(\mathbf{g},\boldsymbol{l}) = \lVert\mathbf{g}_{xyz}\times\boldsymbol{l}_{\mathrm{v}}\rVert\,\mathbf{1}+\lVert\mathbf{g}\rVert_{\circ}\lVert\boldsymbol{l}\rVert_{\circ}$	
Cosine of angle ϕ between lines ***l*** and **k**. $\cos\phi\,(\boldsymbol{l},\mathbf{k}) = (\boldsymbol{l}_{\mathrm{v}}\cdot\mathbf{k}_{\mathrm{v}})\,\mathbf{1}+\lVert\boldsymbol{l}\rVert_{\circ}\lVert\mathbf{k}\rVert_{\circ}$	

Bulk and Weight Duals

- Multiply by metric or antimetric, then take complement

DUALITY

Bulk Dual

$\mathbf{u}^{\bigstar}$

- Defined as $\overline{\mathbf{G}\mathbf{u}}$.
- Gives the dual of the bulk components of **u**.

Weight Dual

$\mathbf{u}^{\medwhitestar}$

- Defined as $\overline{\mathbb{G}\mathbf{u}}$.
- Gives the dual of the weight components of **u**.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{u}^{\bigstar}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0
$\mathbf{u}_{\bigstar}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$\mathbf{e}_4$	0
$\mathbf{u}^{\medwhitestar}$	0	0	0	0	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	0	$\mathbf{1}$
$\mathbf{u}_{\medwhitestar}$	0	0	0	0	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	$\mathbf{1}$

Interior Products

- Two exterior products combined with two duals
- Eight *interior* products using right and left duals

• Bulk contraction	$\mathbf{a} \vee \mathbf{b}^{\bigstar}$	$\mathbf{b}_{\bigstar} \vee \mathbf{a}$
• Weight contraction	$\mathbf{a} \vee \mathbf{b}^{\text{☆}}$	$\mathbf{b}_{\text{☆}} \vee \mathbf{a}$
• Bulk expansion	$\mathbf{a} \wedge \mathbf{b}^{\bigstar}$	$\mathbf{b}_{\bigstar} \wedge \mathbf{a}$
• Weight expansion	$\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$	$\mathbf{b}_{\text{☆}} \wedge \mathbf{a}$

Sec. 2.13

Contraction and Expansion

- Subtract grades or antigrades

Contraction

$$\mathbf{a} \vee \mathbf{b}^{\bigstar}$$

- Decreases the grade of **a** by removing the subspace spanned by **b**.
- The result is a lower-dimensional object that is contained by **a** and is orthogonal to **b**.
- The grade of the result is $\operatorname{gr}(\mathbf{a}) - \operatorname{gr}(\mathbf{b})$.

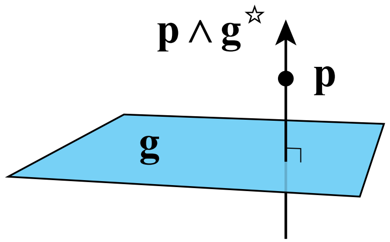
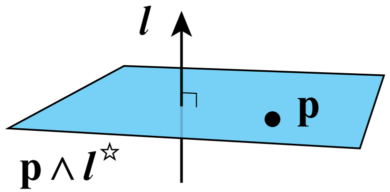
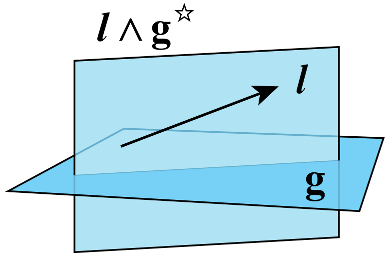
DUALITY

Expansion

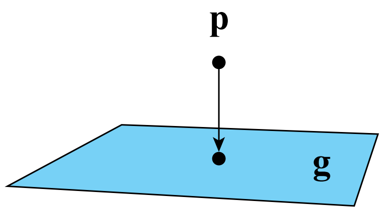
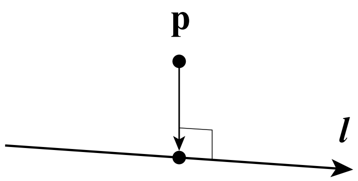
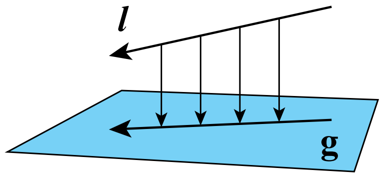
$$\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$$

- Increases the grade of **a** by adding the subspace not spanned by **b**.
- The result is a higher-dimensional object that contains **a** and is orthogonal to **b**.
- The antigrade of the result is $\operatorname{ag}(\mathbf{a}) - \operatorname{ag}(\mathbf{b})$.

Weight Expansion

Expansion Operation	**Illustration**
Line containing point **p** and orthogonal to plane **g**. $\mathbf{p} \wedge \mathbf{g}^{\star} = -p_w g_x \mathbf{e}_{41} - p_w g_y \mathbf{e}_{42} - p_w g_z \mathbf{e}_{43} + (p_z g_y - p_y g_z)\mathbf{e}_{23} + (p_x g_z - p_z g_x)\mathbf{e}_{31} + (p_y g_x - p_x g_y)\mathbf{e}_{12}$	
Plane containing point **p** and orthogonal to line ***l***. $\mathbf{p} \wedge \boldsymbol{l}^{\star} = -p_w l_{vx}\mathbf{e}_{423} - p_w l_{vy}\mathbf{e}_{431} - p_w l_{vz}\mathbf{e}_{412} + (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\mathbf{e}_{321}$	
Plane containing line ***l*** and orthogonal to plane **g**. $\boldsymbol{l} \wedge \mathbf{g}^{\star} = (l_{vy} g_z - l_{vz} g_y)\mathbf{e}_{423} + (l_{vz} g_x - l_{vx} g_z)\mathbf{e}_{431} + (l_{vx} g_y - l_{vy} g_x)\mathbf{e}_{412} - (l_{mx} g_x + l_{my} g_y + l_{mz} g_z)\mathbf{e}_{321}$	

Orthogonal Projection

Projection Operation	**Illustration**
Orthogonal projection of point **p** onto plane **g**. $\mathbf{g} \vee \left(\mathbf{p} \wedge \mathbf{g}^{\star}\right) = \left(g_x^2 + g_y^2 + g_z^2\right)\left(p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4\right)$ $- \left(g_x p_x + g_y p_y + g_z p_z + g_w p_w\right)\left(g_x\mathbf{e}_1 + g_y\mathbf{e}_2 + g_z\mathbf{e}_3\right)$	
Orthogonal projection of point **p** onto line ***l***. $\boldsymbol{l} \vee \left(\mathbf{p} \wedge \boldsymbol{l}^{\star}\right) = \left(l_{vx}p_x + l_{vy}p_y + l_{vz}p_z\right)\left(l_{vx}\mathbf{e}_1 + l_{vy}\mathbf{e}_2 + l_{vz}\mathbf{e}_3\right) + \left(l_{vx}^2 + l_{vy}^2 + l_{vz}^2\right)p_w\mathbf{e}_4$ $+ \left(l_{vy}l_{mz} - l_{vz}l_{my}\right)p_w\mathbf{e}_1 + \left(l_{vz}l_{mx} - l_{vx}l_{mz}\right)p_w\mathbf{e}_2 + \left(l_{vx}l_{my} - l_{vy}l_{mx}\right)p_w\mathbf{e}_3$	
Orthogonal projection of line ***l*** onto plane **g**. $\mathbf{g} \vee \left(\boldsymbol{l} \wedge \mathbf{g}^{\star}\right) = \left(g_x^2 + g_y^2 + g_z^2\right)\left(l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43}\right)$ $- \left(g_x l_{vx} + g_y l_{vy} + g_z l_{vz}\right)\left(g_x\mathbf{e}_{41} + g_y\mathbf{e}_{42} + g_z\mathbf{e}_{43}\right)$ $+ \left(g_x l_{mx} + g_y l_{my} + g_z l_{mz}\right)\left(g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31} + g_z\mathbf{e}_{12}\right)$ $+ \left(g_z l_{vy} - g_y l_{vz}\right)g_w\mathbf{e}_{23} + \left(g_x l_{vz} - g_z l_{vx}\right)g_w\mathbf{e}_{31} + \left(g_y l_{vx} - g_x l_{vy}\right)g_w\mathbf{e}_{12}$	

Sec. 2.13.6

Geometric / Clifford Algebra

- Geometric product $\mathbf{a} ⟑ \mathbf{b}$

- Geometric antiproduct $\mathbf{a} ⟇ \mathbf{b}$

- We use upward and downward wedge with dot inside
- “Wedge-dot” and “Antiwedge-dot”

- G.P. historically denoted by juxtaposition without symbol
- But duality gives us two products that need distinguishing

Geometric Product and Antiproduct

- Vectors square to inner product instead of zero
- Product satisfy the usual De Morgan law

$$\mathbf{a} \mathbin{\dot{\vee}} \mathbf{b} = \overline{\underline{\mathbf{a}} \mathbin{\dot{\wedge}} \underline{\mathbf{b}}}$$

Geometric Product

$\mathbf{a} \mathbin{\dot{\wedge}} \mathbf{b}$

- Vector $\mathbf{v}$ squares to dot product $\mathbf{v} \bullet \mathbf{v}$.
- Includes wedge product $\wedge$.
- Identity is scalar $\mathbf{1}$.

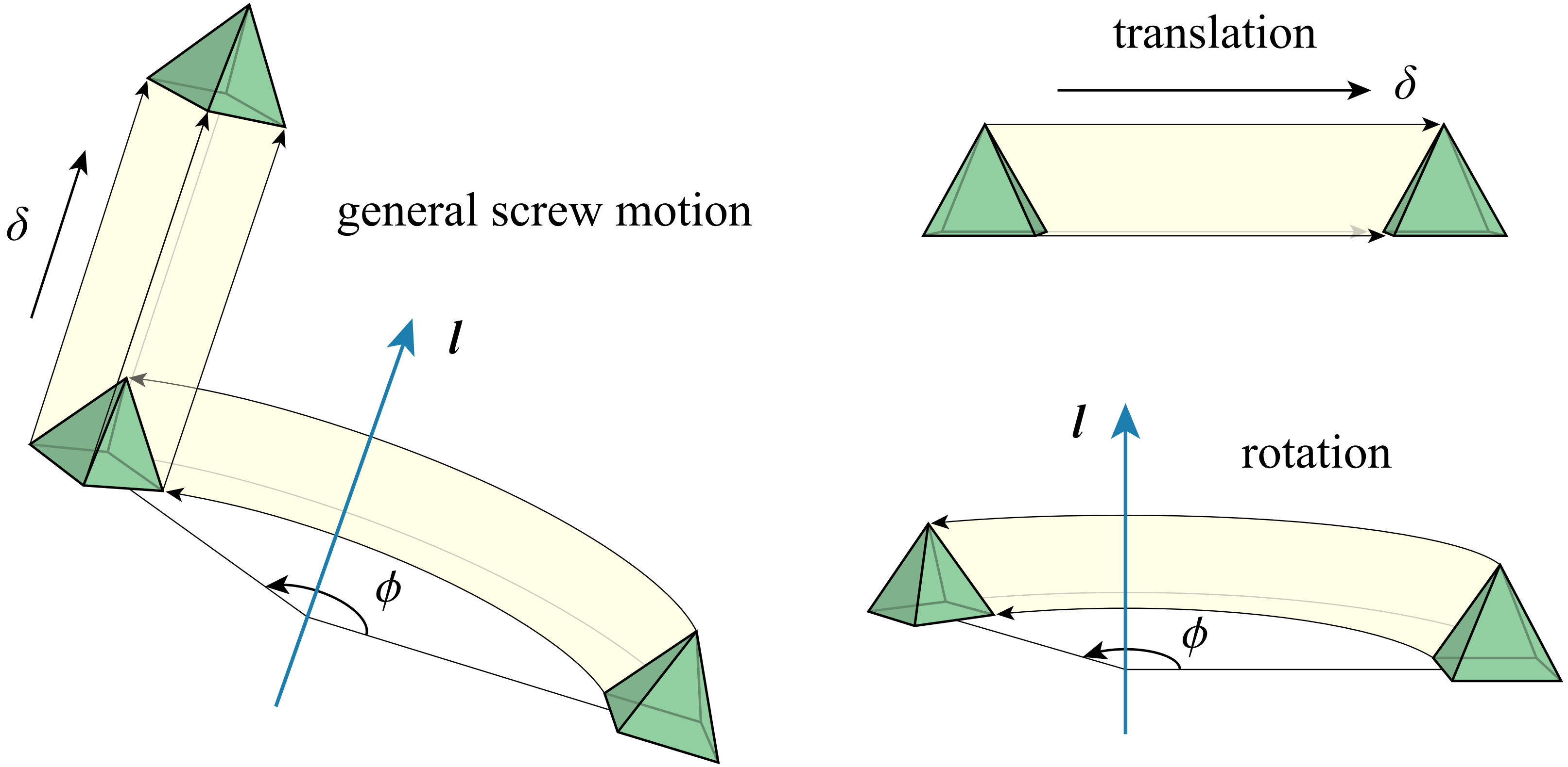
DUALITY

Geometric Antiproduct

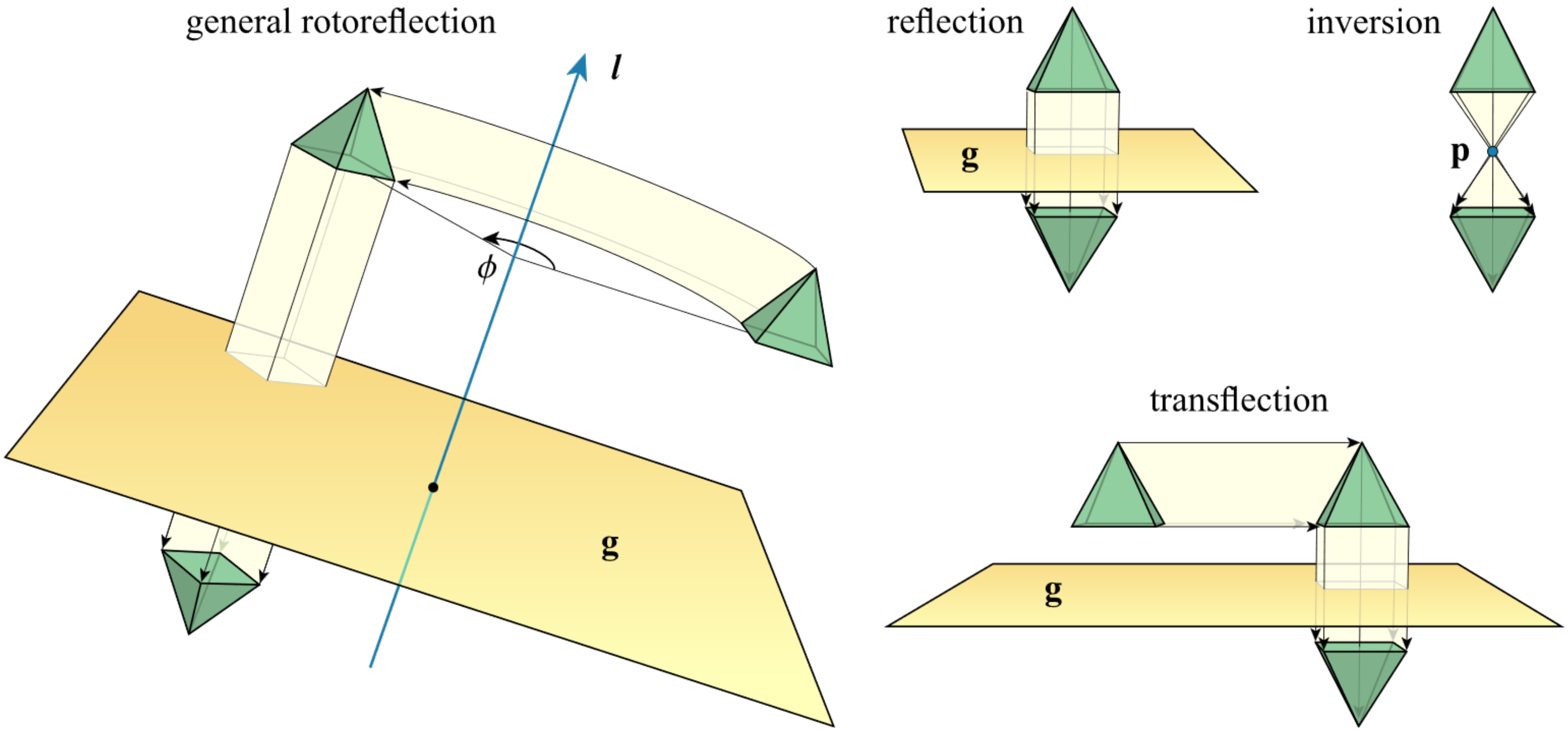
$\mathbf{a} \mathbin{\dot{\vee}} \mathbf{b}$

- Antivector squares to antidot product $\mathbf{v} \circ \mathbf{v}$.
- Includes antiwedge product $\vee$.
- Identity is antiscalar $\mathbb{1}$.

Proper Euclidean Isometries



Improper Euclidean Isometries



Geometric Product

- Geometric **product** in 4D space fixes the origin
- Cannot perform transformations we want

- Geometric **antiproduct** performs Euclidean isometries
- Uses sandwiching similar to quaternions

Plane Reflection

- Sandwich antiproduct with plane **g** performs reflection:

$$\mathbf{u}' = \mathbf{g} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \mathbf{g}$$

- Multiple reflections stack outward from **u**:

$$\mathbf{u}' = \left(\mathbf{h} \mathbin{\dot{\vee}} \mathbf{g}\right) \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \left(\mathbf{g} \mathbin{\dot{\vee}} \mathbf{h}\right)$$

- Basis for all Euclidean isometries

Reverse and Antireverse

- Multiply vector or antivector factors in reverse order

Reverse

$\tilde{\mathbf{u}}$

- Reverses the order in which the vector factors of **u** are multiplied under the wedge product.
- Changes sign when the grade of **u** is 2 or 3 modulo 4.

DUALITY

Antireverse

$\underset{\sim}{\mathbf{u}}$

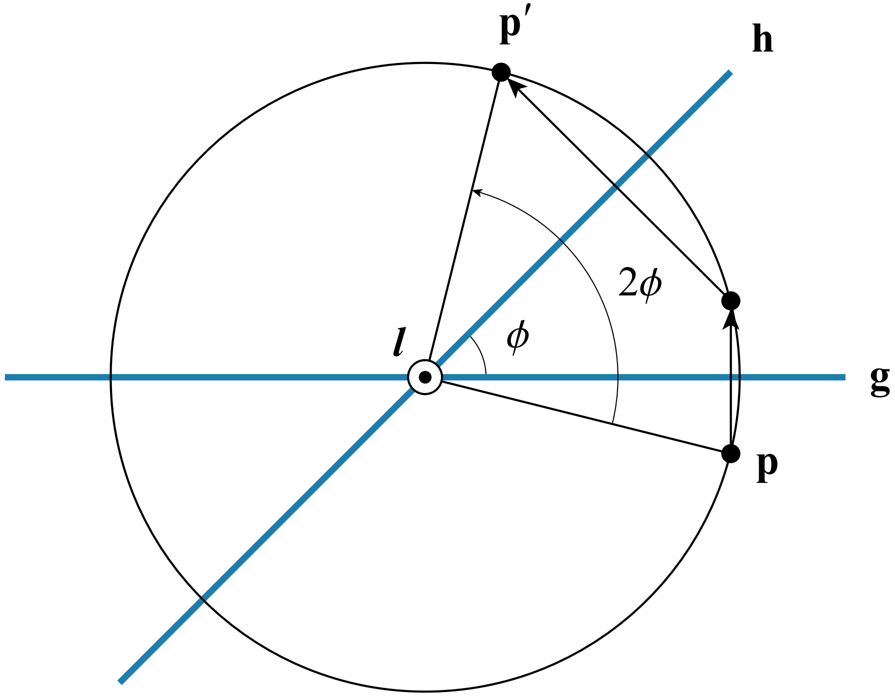
- Reverses the order in which the antivector factors of **u** are multiplied under the antiwedge product.
- Changes sign when the antigrade of **u** is 2 or 3 modulo 4.

$\mathbf{u}$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\tilde{\mathbf{u}}$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$\mathbb{1}$
$\underset{\sim}{\mathbf{u}}$	1	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Rotation about a Line

- Let **g** and **h** be planes meeting at an angle ϕ
- Reflection across **g** followed by **h** is rotation through 2ϕ about line ***l*** where planes intersect

$$\boldsymbol{l} = \frac{\mathbf{h} \vee \mathbf{g}}{\left\| \mathbf{h} \vee \mathbf{g} \right\|_{\circ}}$$



Rotation about a Line

- Planes multiply together under geometric antiproduct to form rotation operator **R**

$$\mathbf{p}' = \mathbf{h} \mathbin{⟇} \left(\mathbf{g} \mathbin{⟇} \mathbf{p} \mathbin{⟇} \mathbf{g} \right) \mathbin{⟇} \mathbf{h}$$

$$\mathbf{p}' = \mathbf{R} \mathbin{⟇} \mathbf{p} \mathbin{⟇} \underset{\sim}{\mathbf{R}}$$

$$\mathbf{R} = \mathbf{h} \mathbin{⟇} \mathbf{g}$$

Rotation about a Line

- General form of rotation operator **R**:

$$\mathbf{R} = \boldsymbol{l} \sin\phi + \mathbb{1} \cos\phi$$

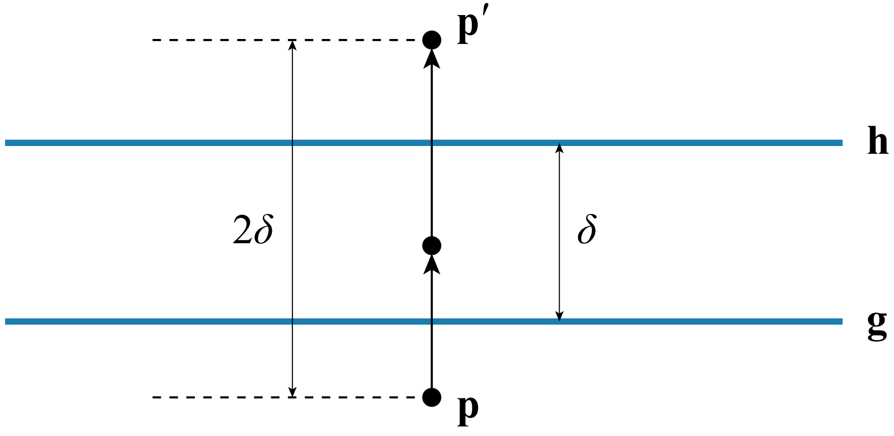
- Rotates through angle 2ϕ about unitized line ***l***

$$\mathbf{u}' = \mathbf{R} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{R}}$$

- Rotates any geometry and even other operators

Translation

- If planes **g** and **h** are parallel, result is a translation
- Translation goes along normal direction by twice the distance δ between the planes



Translation

- General form of translation operator **T**:

$$\mathbf{T} = \tau_x \mathbf{e}_{23} + \tau_y \mathbf{e}_{31} + \tau_z \mathbf{e}_{12} + \mathbb{1}$$

- Translates by displacement vector 2**τ**

$$\mathbf{u}' = \mathbf{T} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{T}}$$

- Translates any geometry and even other operators

Euclidean Isometry Operators

- Sandwiches with geometric antiproduct perform Euclidean isometries
- Motor = MOtion operaTOR
- Flector = reFLECtion operaTOR

Motor

- General form of a motor:

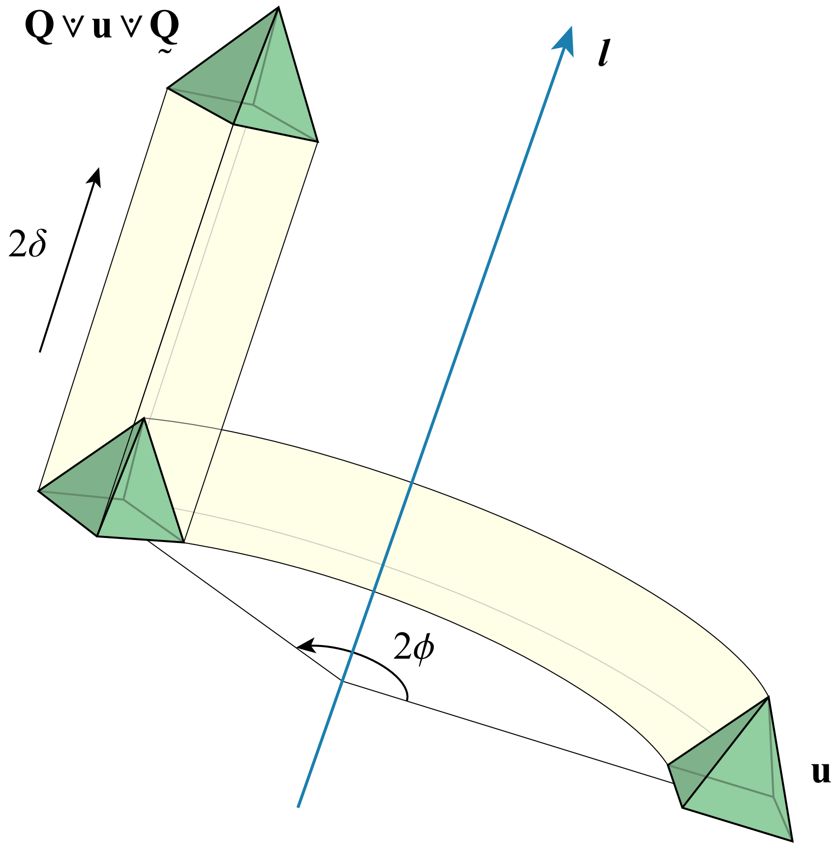
$$\mathbf{Q} = \underbrace{Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1}}_{\text{Rotation Quaternion}} + \underbrace{Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}}_{\text{Moment and Displacement}}$$

- Performs any combination of rotations and translations

$$\mathbf{u}' = \mathbf{Q} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{Q}}$$

Sec. 3.6.1

Motor



$$\mathbf{Q} = \exp_{\dot\vee}\left[\left(\delta\mathbf{1} + \varphi\mathbb{1}\right)\dot\vee\,\boldsymbol{l}\right] = \boldsymbol{l}\sin\varphi - \boldsymbol{l}^{\star}\delta\cos\varphi - \delta\sin\varphi + \mathbb{1}\cos\varphi$$

Motor Parameterization

- A motion operator is parameterized by:
  - A unitized line ***l***
  - A rotation angle ϕ
  - A displacement distance δ

- Exponential with respect to geometric antiproduct:

$$\mathbf{Q} = \exp_{\mathbin{\dot\vee}}\left[\left(\delta\mathbf{1} + \phi\mathbb{1}\right) \mathbin{\dot\vee} \boldsymbol{l}\right] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

- $\delta\mathbf{1} + \phi\mathbb{1}$ is *pitch* of screw transformation

Matrix Advantages

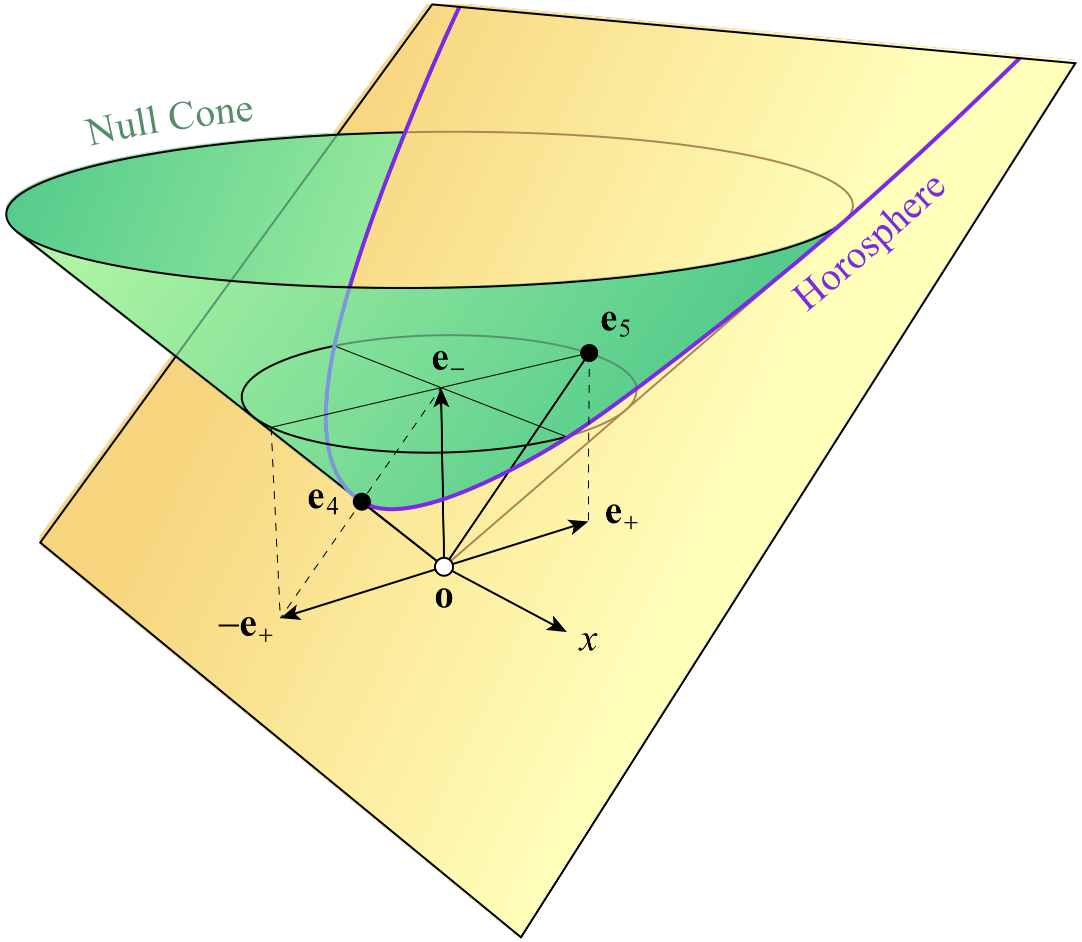
- Can represent more transformations
- Can read off origin and axis directions in transformed space
- Faster to transform objects
- Faster to compose

Motor Advantages

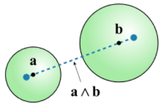
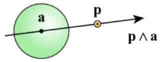
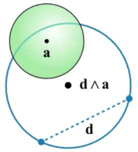
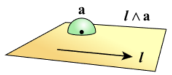
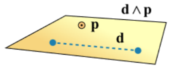
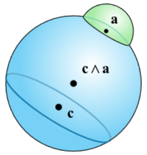
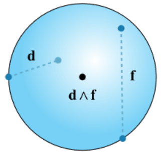
- Smaller storage requirements
    - Usually 8 floats, but can reduce to 6
- Inversion is trivial
    - Just reverse, negating six bivector components
- Better parameterization
- Better interpolation properties

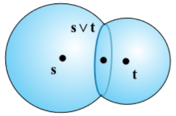
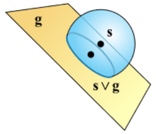
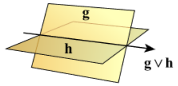
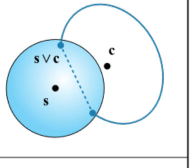
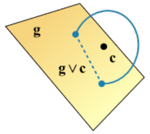
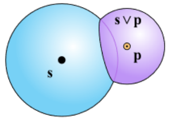
Conformal Algebras

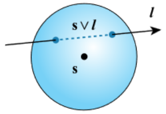
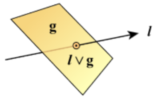
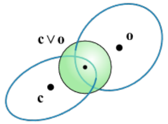
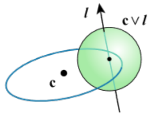
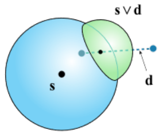
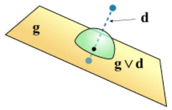
- 5D representation space for 3D geometry and motion
- Doubly projective
- Contains round objects:
  - Spheres
  - Circles
  - Dipoles
  - Round points
- Points, lines, and planes are special cases with infinite radii



Conformal Exterior Algebra

Join Operation	Illustration
Dipole containing round points **a** and **b**. $\mathbf{a}\wedge\mathbf{b} = (a_wb_x - a_xb_w)\,\mathbf{e}_{41} + (a_wb_y - a_yb_w)\,\mathbf{e}_{42} + (a_wb_z - a_zb_w)\,\mathbf{e}_{43}$ $+ (a_yb_z - a_zb_y)\,\mathbf{e}_{23} + (a_zb_x - a_xb_z)\,\mathbf{e}_{31} + (a_xb_y - a_yb_x)\,\mathbf{e}_{12}$ $+ (a_xb_u - a_ub_x)\,\mathbf{e}_{15} + (a_yb_u - a_ub_y)\,\mathbf{e}_{25}$ $+ (a_zb_u - a_ub_z)\,\mathbf{e}_{35} + (a_wb_u - a_ub_w)\,\mathbf{e}_{45}$	
Line containing flat point **p** and round point **a**. $\mathbf{p}\wedge\mathbf{a} = (p_xa_w - p_wa_x)\,\mathbf{e}_{415} + (p_za_y - p_ya_z)\,\mathbf{e}_{235}$ $+ (p_ya_w - p_wa_y)\,\mathbf{e}_{425} + (p_xa_z - p_za_x)\,\mathbf{e}_{315}$ $+ (p_za_w - p_wa_z)\,\mathbf{e}_{435} + (p_ya_x - p_xa_y)\,\mathbf{e}_{125}$	
Circle containing dipole **d** and round point **a**. $\mathbf{d}\wedge\mathbf{a} = (d_{vy}a_z - d_{vz}a_y + d_{mx}a_w)\,\mathbf{e}_{423} + (d_{vz}a_x - d_{vx}a_z + d_{my}a_w)\,\mathbf{e}_{431}$ $+ (d_{vx}a_y - d_{vy}a_x + d_{mz}a_w)\,\mathbf{e}_{412} - (d_{mx}a_x + d_{my}a_y + d_{mz}a_z)\,\mathbf{e}_{321}$ $+ (d_{px}a_w - d_{pw}a_x + d_{vx}a_u)\,\mathbf{e}_{415} + (d_{pz}a_y - d_{py}a_z + d_{mx}a_u)\,\mathbf{e}_{235}$ $+ (d_{py}a_w - d_{pw}a_y + d_{vy}a_u)\,\mathbf{e}_{425} + (d_{px}a_z - d_{pz}a_x + d_{my}a_u)\,\mathbf{e}_{315}$ $+ (d_{pz}a_w - d_{pw}a_z + d_{vz}a_u)\,\mathbf{e}_{435} + (d_{py}a_x - d_{px}a_y + d_{mz}a_u)\,\mathbf{e}_{125}$	
Plane containing line ***l*** and round point **a**. $\boldsymbol{l}\wedge\mathbf{a} = (l_{vz}a_y - l_{vy}a_z - l_{mx}a_w)\,\mathbf{e}_{4235} + (l_{vx}a_z - l_{vz}a_x - l_{my}a_w)\,\mathbf{e}_{4315}$ $+ (l_{vy}a_x - l_{vx}a_y - l_{mz}a_w)\,\mathbf{e}_{4125} + (l_{mx}a_x + l_{my}a_y + l_{mz}a_z)\,\mathbf{e}_{3215}$	
Plane containing dipole **d** and flat point **p**. $\mathbf{d}\wedge\mathbf{p} = (d_{vy}p_z - d_{vz}p_y + d_{mx}p_w)\,\mathbf{e}_{4235}$ $+ (d_{vz}p_x - d_{vx}p_z + d_{my}p_w)\,\mathbf{e}_{4315}$ $+ (d_{vx}p_y - d_{vy}p_x + d_{mz}p_w)\,\mathbf{e}_{4125}$ $- (d_{mx}p_x + d_{my}p_y + d_{mz}p_z)\,\mathbf{e}_{3215}$	
Sphere containing circle **c** and round point **a**. $\mathbf{c}\wedge\mathbf{a} = -(c_{gx}a_x + c_{gy}a_y + c_{gz}a_z + c_{gw}a_w)\,\mathbf{e}_{1234}$ $+ (c_{vz}a_y - c_{vy}a_z + c_{gx}a_u - c_{mx}a_w)\,\mathbf{e}_{4235}$ $+ (c_{vx}a_z - c_{vz}a_x + c_{gy}a_u - c_{my}a_w)\,\mathbf{e}_{4315}$ $+ (c_{vy}a_x - c_{vx}a_y + c_{gz}a_u - c_{mz}a_w)\,\mathbf{e}_{4125}$ $+ (c_{mx}a_x + c_{my}a_y + c_{mz}a_z + c_{gw}a_u)\,\mathbf{e}_{3215}$	
Sphere containing dipoles **d** and **f**. $\mathbf{d}\wedge\mathbf{f} = -(d_{vx}f_{mx} + d_{vy}f_{my} + d_{vz}f_{mz} + d_{mx}f_{vx} + d_{my}f_{vy} + d_{mz}f_{vz})\,\mathbf{e}_{1234}$ $+ (d_{vy}f_{pz} - d_{vz}f_{py} + d_{pz}f_{vy} - d_{py}f_{vz} + d_{mx}f_{pw} + d_{pw}f_{mx})\,\mathbf{e}_{4235}$ $+ (d_{vz}f_{px} - d_{vx}f_{pz} + d_{px}f_{vz} - d_{pz}f_{vx} + d_{my}f_{pw} + d_{pw}f_{my})\,\mathbf{e}_{4315}$ $+ (d_{vx}f_{py} - d_{vy}p_{px} + d_{py}f_{vx} - d_{px}f_{vy} + d_{mz}f_{pw} + d_{pw}f_{mz})\,\mathbf{e}_{4125}$ $- (d_{mx}f_{px} + d_{my}f_{py} + d_{mz}f_{pz} + d_{px}f_{mx} + d_{py}f_{my} + d_{pz}f_{mz})\,\mathbf{e}_{3215}$	

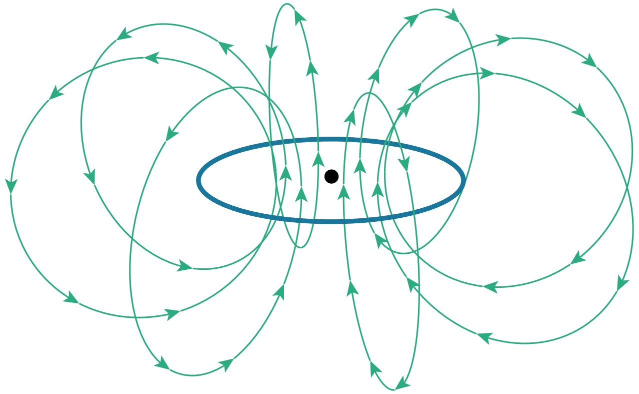
Meet Operation	Illustration
Circle where spheres **s** and **t** intersect. $\mathbf{s}\vee\mathbf{t} = (s_ut_x - s_xt_u)\,\mathbf{e}_{423} + (s_ut_y - s_yt_u)\,\mathbf{e}_{431}$ $+ (s_ut_z - s_zt_u)\,\mathbf{e}_{412} + (s_ut_w - s_wt_u)\,\mathbf{e}_{321}$ $+ (s_zt_y - s_yt_z)\,\mathbf{e}_{415} + (s_xt_z - s_zt_x)\,\mathbf{e}_{425} + (s_yt_x - s_xt_y)\,\mathbf{e}_{435}$ $+ (s_xt_w - s_wt_x)\,\mathbf{e}_{235} + (s_yt_w - s_wt_y)\,\mathbf{e}_{315} + (s_zt_w - s_wt_z)\,\mathbf{e}_{125}$	
Circle where sphere **s** and plane **g** intersect. $\mathbf{s}\vee\mathbf{g} = s_ug_x\,\mathbf{e}_{423} + s_ug_y\,\mathbf{e}_{431} + s_ug_z\,\mathbf{e}_{412} + s_ug_w\,\mathbf{e}_{321}$ $+ (s_zg_y - s_yg_z)\,\mathbf{e}_{415} + (s_xg_z - s_zg_x)\,\mathbf{e}_{425} + (s_yg_x - s_xg_y)\,\mathbf{e}_{435}$ $+ (s_xg_w - s_wg_x)\,\mathbf{e}_{235} + (s_yg_w - s_wg_y)\,\mathbf{e}_{315} + (s_zg_w - s_wg_z)\,\mathbf{e}_{125}$	
Line where planes **g** and **h** intersect. $\mathbf{g}\vee\mathbf{h} = (g_zh_y - g_yh_z)\,\mathbf{e}_{415} + (g_xh_w - g_wh_x)\,\mathbf{e}_{235}$ $+ (g_xh_z - g_zh_x)\,\mathbf{e}_{425} + (g_yh_w - g_wh_y)\,\mathbf{e}_{315}$ $+ (g_yh_x - g_xh_y)\,\mathbf{e}_{435} + (g_zh_w - g_wh_z)\,\mathbf{e}_{125}$	
Dipole where sphere **s** and circle **c** intersect. $\mathbf{s}\vee\mathbf{c} = (s_yc_{gz} - s_zc_{gy} + s_uc_{vx})\,\mathbf{e}_{41} + (s_wc_{gx} - s_xc_{gw} + s_uc_{mx})\,\mathbf{e}_{23}$ $+ (s_zc_{gx} - s_xc_{gz} + s_uc_{vy})\,\mathbf{e}_{42} + (s_wc_{gy} - s_yc_{gw} + s_uc_{my})\,\mathbf{e}_{31}$ $+ (s_xc_{gy} - s_yc_{gx} + s_uc_{vz})\,\mathbf{e}_{43} + (s_wc_{gz} - s_zc_{gw} + s_uc_{mz})\,\mathbf{e}_{12}$ $+ (s_zc_{my} - s_yc_{mz} + s_wc_{vx})\,\mathbf{e}_{15} + (s_xc_{mz} - s_zc_{mx} + s_wc_{vy})\,\mathbf{e}_{25}$ $+ (s_yc_{mx} - s_xc_{my} + s_wc_{vz})\,\mathbf{e}_{35} - (s_xc_{vx} + s_yc_{vy} + s_zc_{vz})\,\mathbf{e}_{45}$	
Dipole where plane **g** and circle **c** intersect. $\mathbf{g}\vee\mathbf{c} = (g_yc_{gz} - g_zc_{gy})\,\mathbf{e}_{41} + (g_wc_{gx} - g_xc_{gw})\,\mathbf{e}_{23}$ $+ (g_zc_{gx} - g_xc_{gz})\,\mathbf{e}_{42} + (g_wc_{gy} - g_yc_{gw})\,\mathbf{e}_{31}$ $+ (g_xc_{gy} - g_yc_{gx})\,\mathbf{e}_{43} + (g_wc_{gz} - g_zc_{gw})\,\mathbf{e}_{12}$ $+ (g_zc_{my} - g_yc_{mz} + g_wc_{vx})\,\mathbf{e}_{15} + (g_xc_{mz} - g_zc_{mx} + g_wc_{vy})\,\mathbf{e}_{25}$ $+ (g_yc_{mx} - g_xc_{my} + g_wc_{vz})\,\mathbf{e}_{35} - (g_xc_{vx} + g_yc_{vy} + g_zc_{vz})\,\mathbf{e}_{45}$	
Round point centered at flat point **p** and contained by sphere **s**. $\mathbf{s}\vee\mathbf{p} = s_up_x\,\mathbf{e}_1 + s_up_y\,\mathbf{e}_2 + s_up_z\,\mathbf{e}_3 + s_up_w\,\mathbf{e}_4$ $- (s_xp_x + s_yp_y + s_zp_z + s_wp_w)\,\mathbf{e}_5$	

Meet Operation	Illustration
Dipole where sphere **s** and line ***l*** intersect. $\mathbf{s}\vee\boldsymbol{l} = s_ul_{vx}\,\mathbf{e}_{41} + s_ul_{vy}\,\mathbf{e}_{42} + s_ul_{vz}\,\mathbf{e}_{43}$ $+ s_ul_{mx}\,\mathbf{e}_{23} + s_ul_{my}\,\mathbf{e}_{31} + s_ul_{mz}\,\mathbf{e}_{12}$ $+ (s_zl_{my} - s_yl_{mz} + s_wl_{vx})\,\mathbf{e}_{15} + (s_xl_{mz} - s_zl_{mx} + s_wl_{vy})\,\mathbf{e}_{25}$ $+ (s_yl_{mx} - s_xl_{my} + s_wl_{vz})\,\mathbf{e}_{35} - (s_xl_{vx} + s_yl_{vy} + s_zl_{vz})\,\mathbf{e}_{45}$	
Flat point where plane **g** and line ***l*** intersect. $\mathbf{g}\vee\boldsymbol{l} = (g_zl_{my} - g_yl_{mz} + g_wl_{vx})\,\mathbf{e}_{15} + (g_xl_{mz} - g_zl_{mx} + g_wl_{vy})\,\mathbf{e}_{25}$ $+ (g_yl_{mx} - g_xl_{my} + g_wl_{vz})\,\mathbf{e}_{35} - (g_xl_{vx} + g_yl_{vy} + g_zl_{vz})\,\mathbf{e}_{45}$	
Round point contained by circles **c** and **o**. $\mathbf{c}\vee\mathbf{o} = (c_{gz}o_{my} - c_{gy}o_{mz} + c_{my}o_{gz} - c_{mz}o_{gy} + c_{vx}o_{gw} + c_{gw}o_{vx})\,\mathbf{e}_1$ $+ (c_{gx}o_{mz} - c_{gz}o_{mx} + c_{mz}o_{gx} - c_{mx}o_{gz} + c_{vy}o_{gw} + c_{gw}o_{vy})\,\mathbf{e}_2$ $+ (c_{gy}o_{mx} - c_{gx}o_{my} + c_{mx}o_{gy} - c_{my}o_{gx} + c_{vz}o_{gw} + c_{gw}o_{vz})\,\mathbf{e}_3$ $- (c_{gx}o_{vx} + c_{gy}o_{vy} + c_{gz}o_{vz} + c_{vx}o_{gx} + c_{vy}o_{gy} + c_{vz}o_{gz})\,\mathbf{e}_4$ $- (c_{mx}o_{vx} + c_{my}o_{vy} + c_{mz}o_{vz} + c_{vx}o_{mx} + c_{vy}o_{my} + c_{vz}o_{mz})\,\mathbf{e}_5$	
Round point centered on line ***l*** and contained by circle **c**. $\mathbf{c}\vee\boldsymbol{l} = (c_{gz}l_{my} - c_{gy}l_{mz} + c_{gw}l_{vx})\,\mathbf{e}_1 + (c_{gx}l_{mz} - c_{gz}l_{mx} + c_{gw}l_{vy})\,\mathbf{e}_2$ $+ (c_{gy}l_{mx} - c_{gx}l_{my} + c_{gw}l_{vz})\,\mathbf{e}_3 - (c_{gx}l_{vx} + c_{gy}l_{vy} + c_{gz}l_{vz})\,\mathbf{e}_4$ $- (c_{mx}l_{vx} + c_{my}l_{vy} + c_{mz}l_{vz} + c_{vx}l_{mx} + c_{vy}l_{my} + c_{vz}l_{mz})\,\mathbf{e}_5$	
Round point contained by sphere **s** and dipole **d**. $\mathbf{s}\vee\mathbf{d} = (s_yd_{mz} - s_zd_{my} - s_wd_{vx} + s_ud_{px})\,\mathbf{e}_1$ $+ (s_zd_{mx} - s_xd_{mz} - s_wd_{vy} + s_ud_{py})\,\mathbf{e}_2$ $+ (s_xd_{my} - s_yd_{mx} - s_wd_{vz} + s_ud_{pz})\,\mathbf{e}_3$ $+ (s_xd_{vx} + s_yd_{vy} + s_zd_{vz} + s_ud_{pw})\,\mathbf{e}_4$ $- (s_xd_{px} + s_yd_{py} + s_zd_{pz} + s_wd_{pw})\,\mathbf{e}_5$	
Round point centered in plane **g** and contained by dipole **d**. $\mathbf{g}\vee\mathbf{d} = (g_yd_{mz} - g_zd_{my} - g_wd_{vx})\,\mathbf{e}_1$ $+ (g_zd_{mx} - g_xd_{mz} - g_wd_{vy})\,\mathbf{e}_2$ $+ (g_xd_{my} - g_yd_{mx} - g_wd_{vz})\,\mathbf{e}_3$ $+ (g_xd_{vx} + g_yd_{vy} + g_zd_{vz})\,\mathbf{e}_4$ $- (g_xd_{px} + g_yd_{py} + g_zd_{pz} + g_wd_{pw})\,\mathbf{e}_5$	

Conformal Geometric Algebra

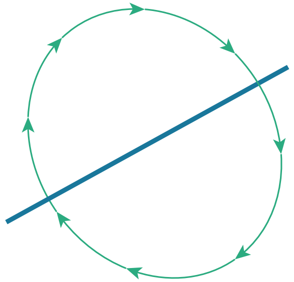
Real Circle / Elliptic Rotation

$\mathbf{R} = \mathbf{c}\sin\phi + \mathbb{1}\cos\phi$



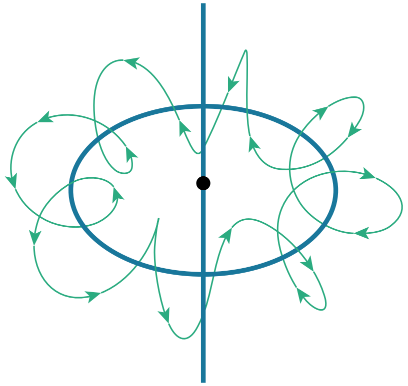
Flat Line / Rotation

$\mathbf{R} = \boldsymbol{l}\sin\phi + \mathbb{1}\cos\phi$



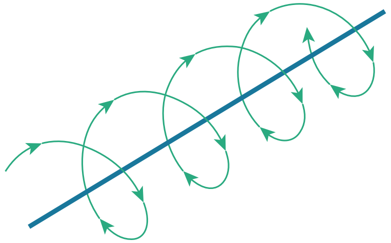
Real Circle + Line

Twisted Elliptic Rotation



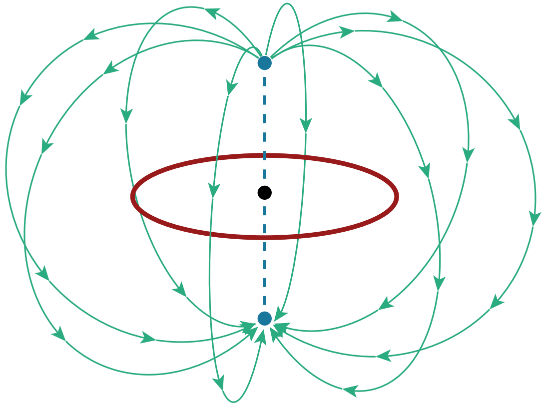
Line or Point in Horizon + Line

Twisted Rotation / Screw Motion



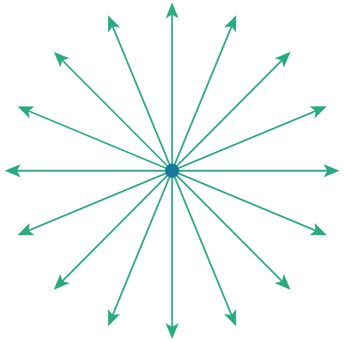
Imaginary Circle / Hyperbolic Rotation

$\mathbf{R} = \mathbf{c}\sinh\phi + \mathbb{1}\cosh\phi$



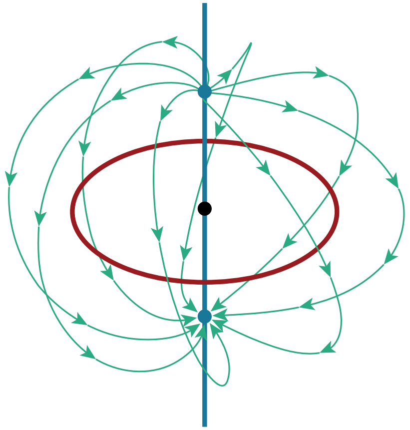
Dual Flat Point / Dilation

$\mathbf{D} = \frac{1-\sigma}{1+\sigma}\mathbf{p}^{\star} + \mathbb{1}$



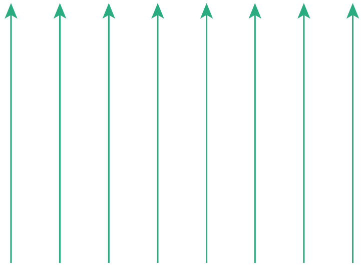
Imaginary Circle + Line

Twisted Hyperbolic Rotation



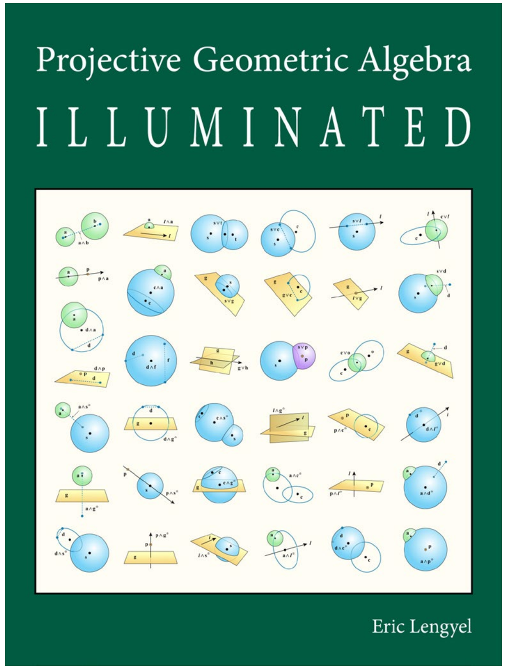
Line or Point in Horizon /Translation

$\mathbf{T} = \mathbf{v}^{\star} + \mathbb{1}$



References

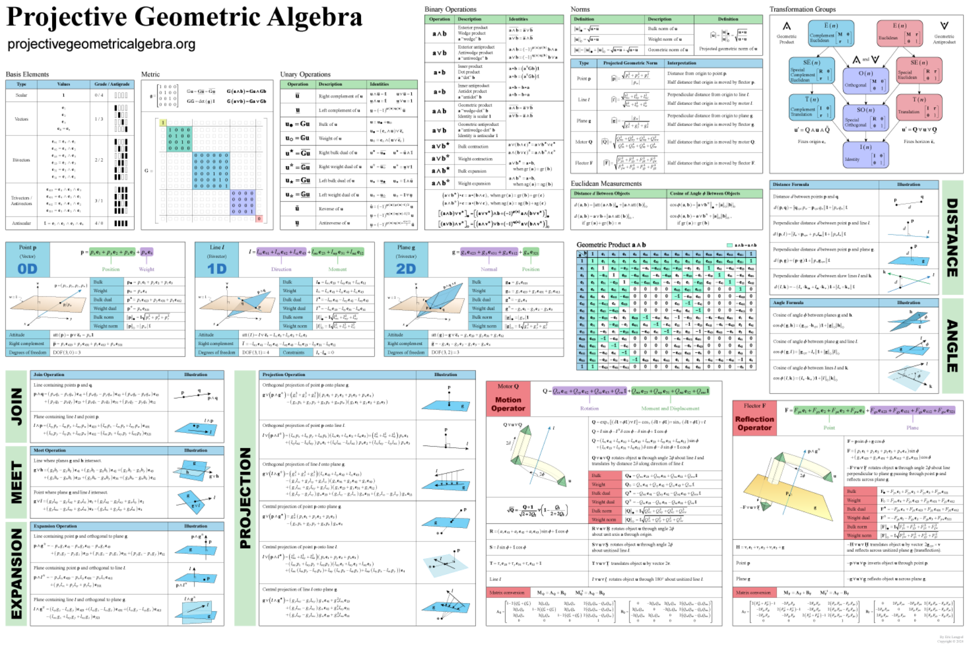
- Projective Geometric Algebra Illuminated
- projectivegeometricalgebra.org



Projective Geometric Algebra

ILLUMINATED

Eric Lengyel



Projective Geometric Algebra

projectivegeometricalgebra.org

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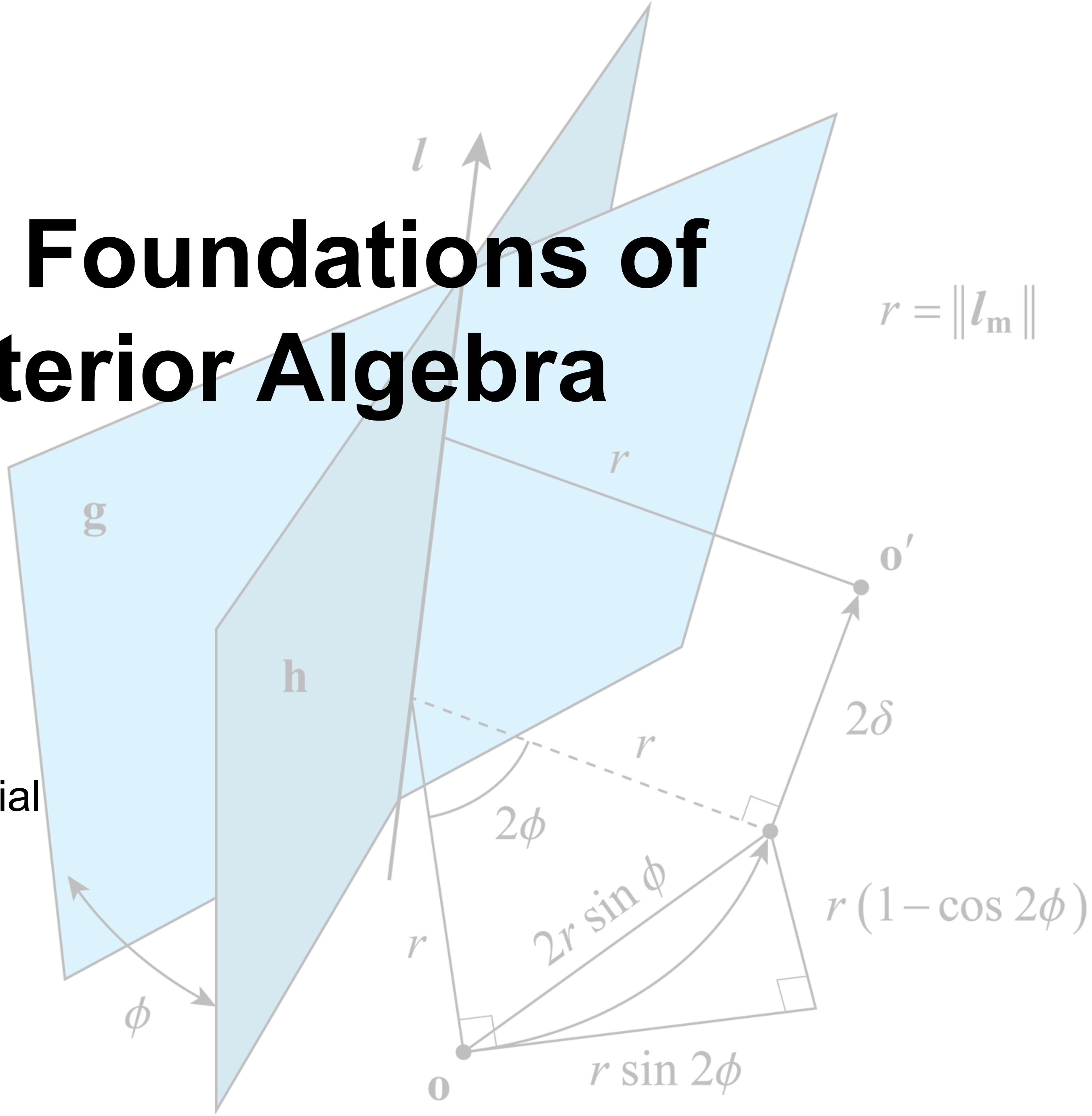
Mathematical Foundations of Projective Exterior Algebra



Eric Lengyel, Ph.D.

Space Imaging Workshop Tutorial
Georgia Tech
October 7, 2024

$\mathbf{e}_{1234}$

4D Exterior Algebra

- One scalar **1**
- Four vector basis elements
- Six bivector basis elements
- Four trivector basis elements
- One antiscalar $\mathbb{1}$

Type	Values	Grade / Antigrade	
Scalar	$\mathbf{1}$	0 / 4	▯▯▯▯
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4 = \mathbf{e}_n$	1 / 3	▮▯▯▯ ▯▮▯▯ ▯▯▮▯ ▯▯▯▮
Bivectors	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$	2 / 2	▮▯▯▮ ▯▮▯▮ ▯▯▮▮ ▯▮▮▯ ▮▯▮▯ ▮▮▯▯
Trivectors / Antivectors	$\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$	3 / 1	▯▮▮▮ ▮▯▮▮ ▮▮▯▮ ▮▮▮▯
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	▮▮▮▮

4D Exterior Product

Wedge Product $\mathbf{a} \wedge \mathbf{b}$

$\mathbf{b}$ $\mathbf{a}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_{1}$	$\mathbf{e}_{1}$	0	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	0	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0	0
$\mathbf{e}_{2}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	0	$-\mathbf{e}_{423}$	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0
$\mathbf{e}_{3}$	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	0	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	0	0	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0
$\mathbf{e}_{4}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	0	$\mathbf{e}_{423}$	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	0	0	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	0	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}$	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

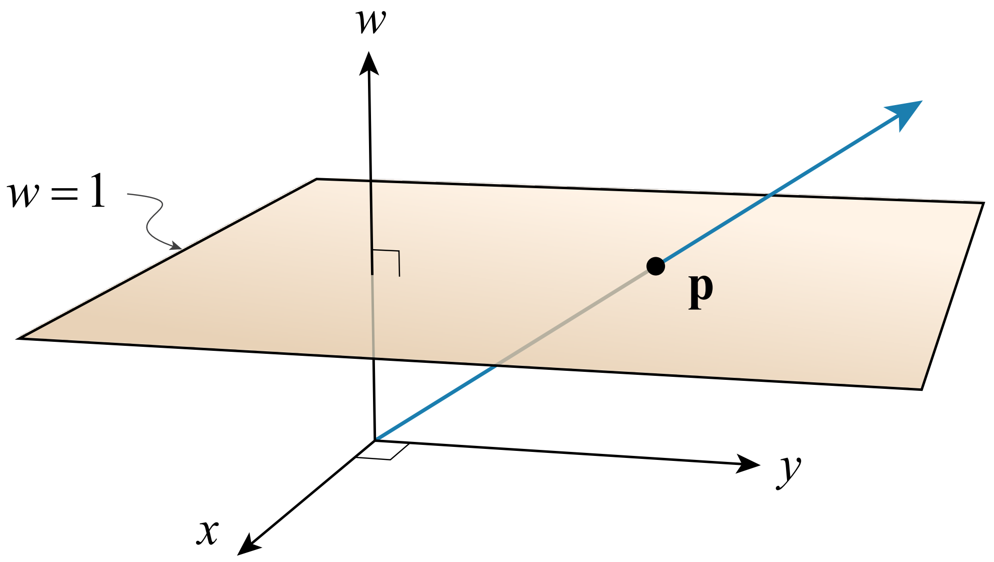
4D Exterior Antiproduct

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

$\mathbf{a}$ \ $\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	$\mathbf{e}_3$
$\mathbf{e}_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	$\mathbf{e}_4$
$\mathbf{e}_{41}$	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	0	0	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	0	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	0	0	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	0	$-\mathbf{1}$	0	0	0	0	0	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	0	0	$-\mathbf{1}$	0	0	0	0	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	0	0	0	$-\mathbf{e}_3$	$\mathbf{e}_2$	0	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	0	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_3$	0	$-\mathbf{e}_1$	$\mathbf{e}_{43}$	0	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	0	0	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$-\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Point

$$\mathbf{p} = \underbrace{p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3}_{\text{Position}} + \underbrace{p_w\,\mathbf{e}_4}_{\text{Weight}}$$



$$\mathbf{p}_\bullet = p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3$$

$$\mathbf{p}_\circ = p_w\,\mathbf{e}_4$$

Special Points

- The origin is simply the point $\mathbf{e}_4$

- Point with zero weight lies at infinity in (x, y, z) direction

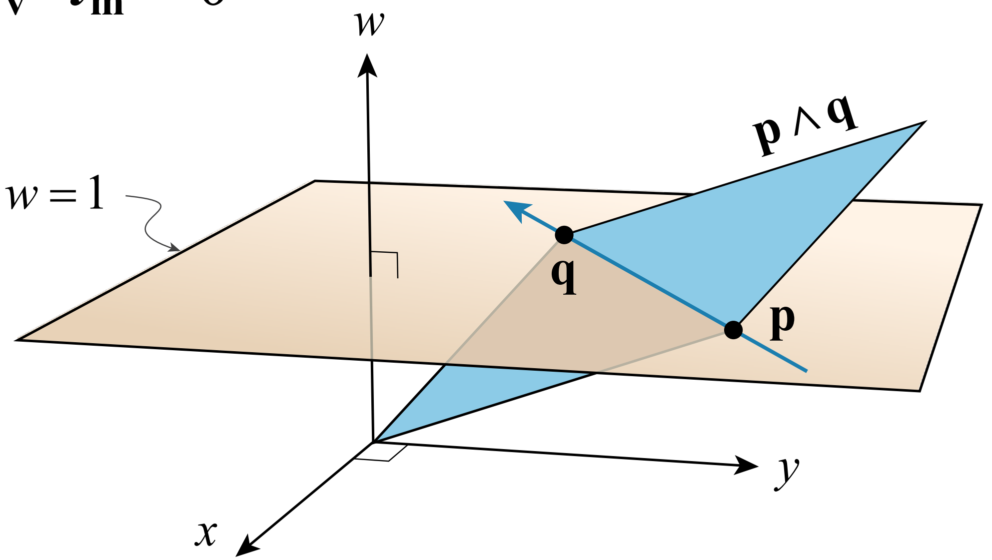
- Points at infinity in opposite directions are equivalent

Line

$$\mathbf{p} \wedge \mathbf{q} = (q_x p_w - p_x q_w)\,\mathbf{e}_{41} + (q_y p_w - p_y q_w)\,\mathbf{e}_{42} + (q_z p_w - p_z q_w)\,\mathbf{e}_{43}$$
$$+ (p_y q_z - p_z q_y)\,\mathbf{e}_{23} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}$$

$$\boldsymbol{l} = \underbrace{l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}}_{\text{Moment}}$$

$$\boldsymbol{l}_{\mathbf{v}} \cdot \boldsymbol{l}_{\mathbf{m}} = 0$$

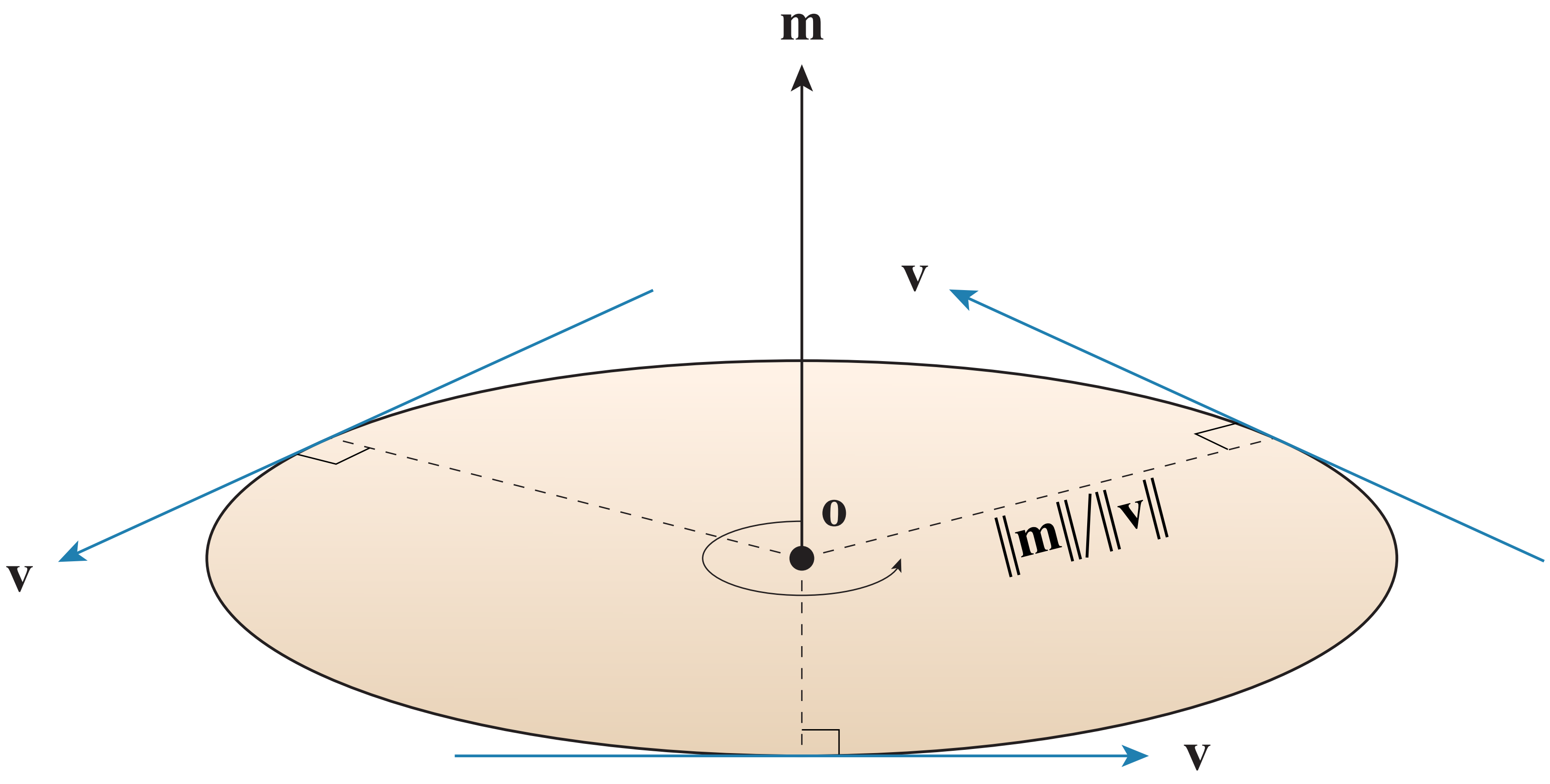


$$\boldsymbol{l}_{\bullet} = l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$$

$$\boldsymbol{l}_{\circ} = l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}$$

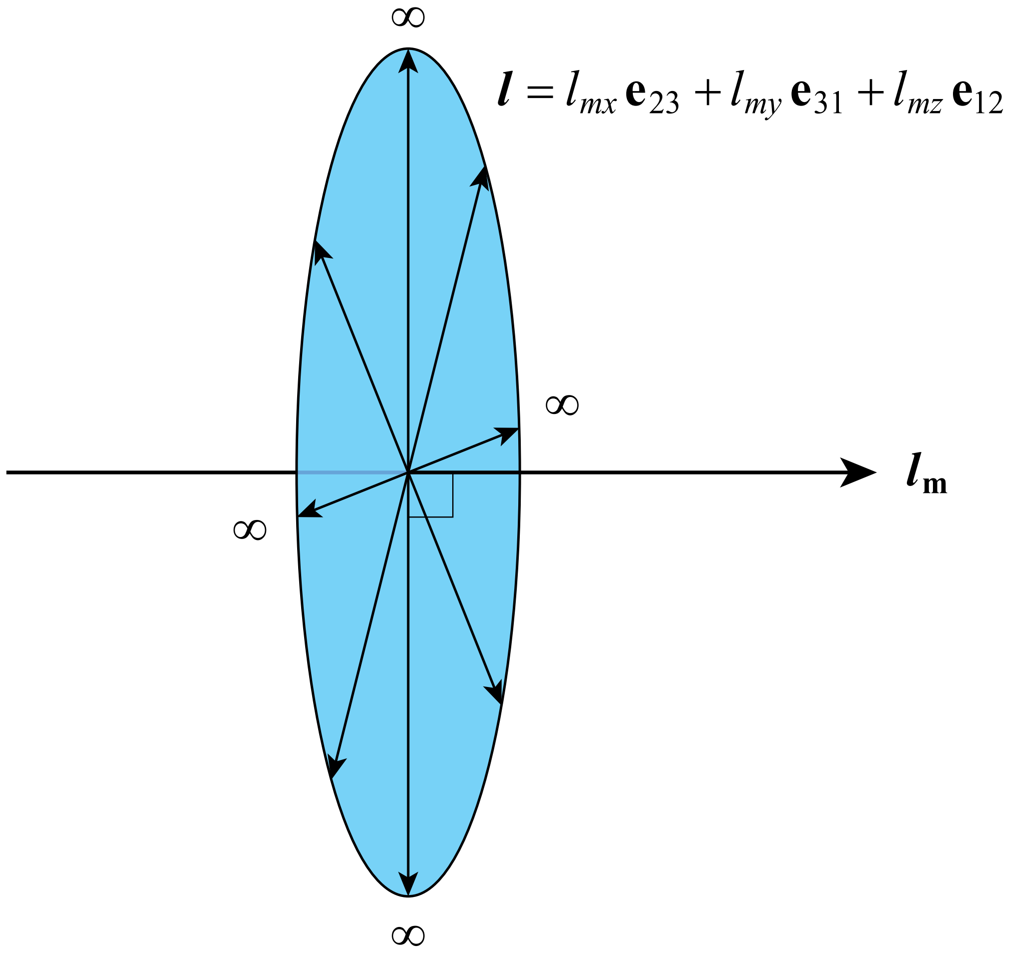
Line Moment

- Contains position information



Lines at Infinity

- Line with zero direction lies at infinity



$$\boldsymbol{l} = l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$$

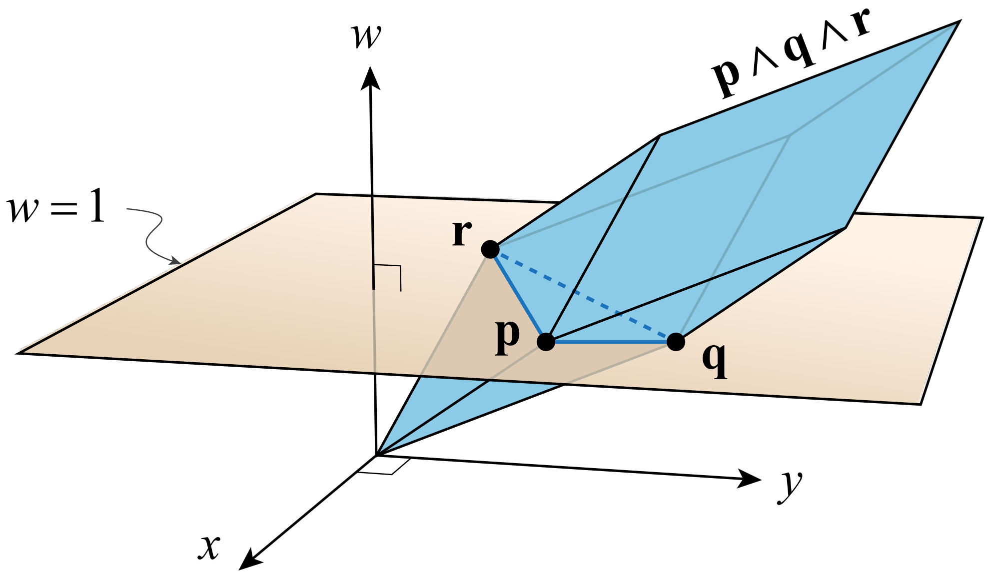
Plane

$$\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx})\,\mathbf{e}_{423} + (l_{vz} p_x - l_{vx} p_z + l_{my})\,\mathbf{e}_{431}$$
$$+ (l_{vx} p_y - l_{vy} p_x + l_{mz})\,\mathbf{e}_{412} - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\,\mathbf{e}_{321}$$

$$\mathbf{g} = \underbrace{g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w \mathbf{e}_{321}}_{\text{Position}}$$

$$\mathbf{g}_\bullet = g_w \mathbf{e}_{321}$$

$$\mathbf{g}_\circ = g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}$$



Sec. 2.4.3

Horizon

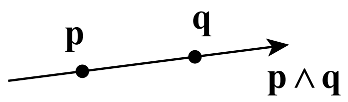
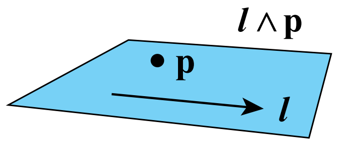
- Plane with zero normal lies at infinity $g_w \mathbf{e}_{321}$
- Contains all points at infinity, all lines at infinity
- Given special name *horizon*
- Complement of origin

Bulk and Weight

- Bulk contains positional information
- Weight contains directional information
- If the bulk is zero, then the object contains the origin
- If the weight zero, then the horizon contains the object

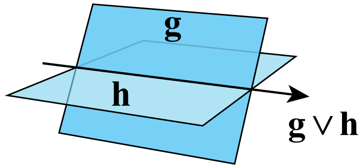
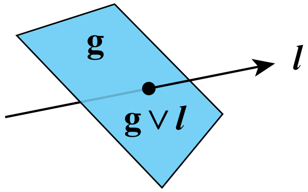
Join

- Wedge product performs join operation
- Produces higher-dimensional object containing both operands

Join Operation	**Illustration**
Line containing points **p** and **q**. $\mathbf{p} \wedge \mathbf{q} = (p_w q_x - p_x q_w)\mathbf{e}_{41} + (p_w q_y - p_y q_w)\mathbf{e}_{42} + (p_w q_z - p_z q_w)\mathbf{e}_{43}$ $+ (p_y q_z - p_z q_y)\mathbf{e}_{23} + (p_z q_x - p_x q_z)\mathbf{e}_{31} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$	
Plane containing line ***l*** and point **p**. $\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx} p_w)\mathbf{e}_{423} + (l_{vz} p_x - l_{vx} p_z + l_{my} p_w)\mathbf{e}_{431}$ $+ (l_{vx} p_y - l_{vy} p_x + l_{mz} p_w)\mathbf{e}_{412} - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\mathbf{e}_{321}$	

Meet

- Antiwedge product performs meet operation
- Produces lower-dimensional object at intersection of operands

Meet Operation	**Illustration**
Line where planes **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\,\mathbf{e}_{41} + (g_x h_z - g_z h_x)\,\mathbf{e}_{42} + (g_y h_x - g_x h_y)\,\mathbf{e}_{43}$ $+ (g_x h_w - g_w h_x)\,\mathbf{e}_{23} + (g_y h_w - g_w h_y)\,\mathbf{e}_{31} + (g_z h_w - g_w h_z)\,\mathbf{e}_{12}$	
Point where plane **g** and line ***l*** intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\,\mathbf{e}_1 + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\,\mathbf{e}_2$ $+ (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\,\mathbf{e}_3 - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\,\mathbf{e}_4$	

Exomorphisms

- Given an *n* x *n* linear transformation **m** that operates on vectors

- The exomorphism **M** is the 2^n x 2^n matrix that operates on the whole algebra

- Exomorphism preserves structure under the wedge product:

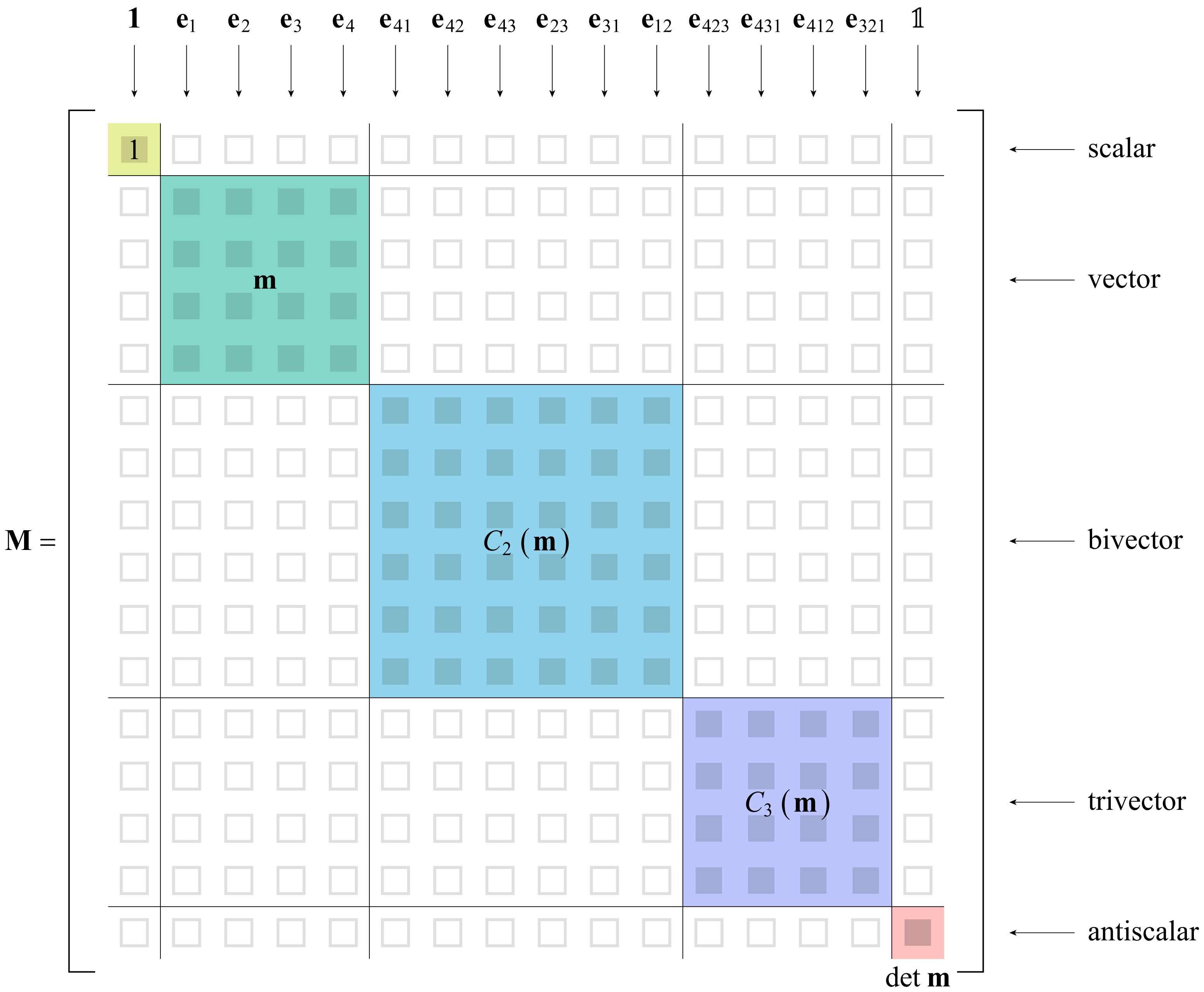
$$\mathbf{M}(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{Ma}) \wedge (\mathbf{Mb})$$

Exomorphisms

- Matrix **M** is block diagonal

- Each block has columns given by wedge products of columns of the original matrix **m**

- These are called *compound matrices* of **m**



$\mathbf{1}$ $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$ $\mathbf{e}_{41}$ $\mathbf{e}_{42}$ $\mathbf{e}_{43}$ $\mathbf{e}_{23}$ $\mathbf{e}_{31}$ $\mathbf{e}_{12}$ $\mathbf{e}_{423}$ $\mathbf{e}_{431}$ $\mathbf{e}_{412}$ $\mathbf{e}_{321}$ $\mathbb{1}$

$\mathbf{M} =$

1

$\mathbf{m}$

$C_2(\mathbf{m})$

$C_3(\mathbf{m})$

$\det \mathbf{m}$

scalar

vector

bivector

trivector

antiscalar

Translation Exomorphism

$$\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -t_z & t_y & 1 & 0 & 0 \\ t_z & 0 & -t_x & 0 & 1 & 0 \\ -t_y & t_x & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_x & -t_y & -t_z & 1 \end{bmatrix}$$

Nonuniform Scale Exomorphism

$$\mathbf{m} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2(\mathbf{m}) = \begin{bmatrix} s_x & 0 & 0 & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 & 0 & 0 \\ 0 & 0 & s_z & 0 & 0 & 0 \\ 0 & 0 & 0 & s_y s_z & 0 & 0 \\ 0 & 0 & 0 & 0 & s_z s_x & 0 \\ 0 & 0 & 0 & 0 & 0 & s_x s_y \end{bmatrix}$$

$$C_3(\mathbf{m}) = \begin{bmatrix} s_y s_z & 0 & 0 & 0 \\ 0 & s_z s_x & 0 & 0 \\ 0 & 0 & s_x s_y & 0 \\ 0 & 0 & 0 & s_x s_y s_z \end{bmatrix}$$

The Metric Tensor

- *n* x *n* matrix that defines dot products of vectors

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$

$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_4 \cdot \mathbf{e}_4 = 0$$

$$\mathfrak{g}_{ij} \equiv \mathbf{v}_i \cdot \mathbf{v}_j$$

Metric Exomorphism

- The metric tensor is a linear transformation
- Thus, it can be extended to a full exomorphism matrix **G**
- There is also a metric *antiexomorphism*, or just “antimetric”, that satisfies

$$\mathbb{G}\mathbf{u} = \underline{\mathbf{G}\overline{\mathbf{u}}} = \overline{\mathbf{G}\underline{\mathbf{u}}}$$

Sec. 2.8.2

Metric and Antimetric

$$\mathbf{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\hline
\square & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
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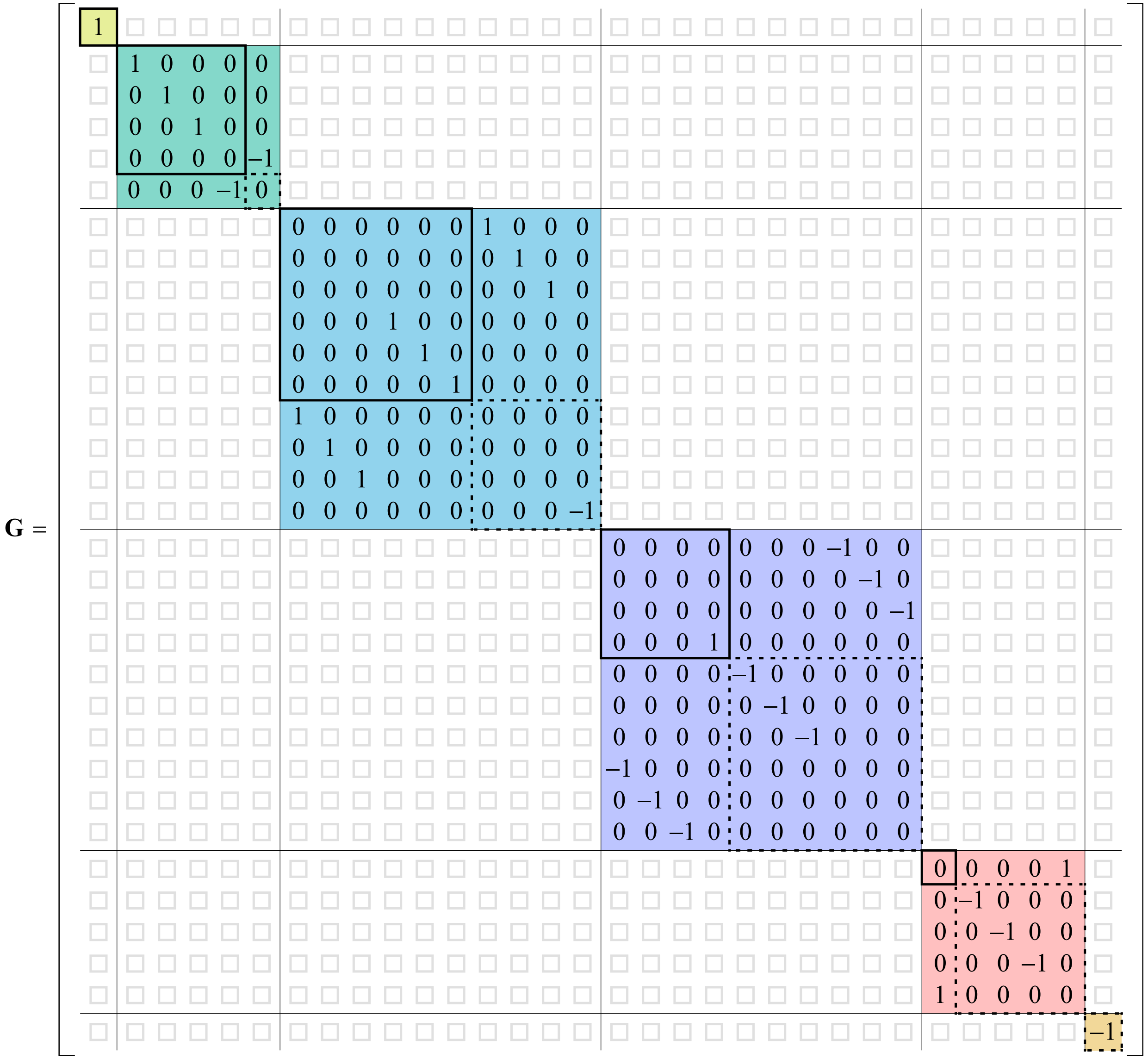
Conformal Metric



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Inner Product

- Dot product defined by metric: $\mathbf{a} \bullet \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}} \mathbf{G} \mathbf{b}\right) \mathbf{1}$
- Antidot product defined by antimetric: $\mathbf{a} \circ \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b}\right) \mathbb{1}$
- Satisfies De Morgan law: $\mathbf{a} \circ \mathbf{b} = \overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}}$

Bulk and Weight Norms

- Two dot products induce two norms
- Bulk norm: $\left\| \mathbf{u} \right\|_{\bullet} = \sqrt{\mathbf{u} \bullet \mathbf{u}}$
- Weight norm: $\left\| \mathbf{u} \right\|_{\circ} = \sqrt{\mathbf{u} \circ \mathbf{u}}$

Bulk and Weight Norms

Type	**Bulk Norm**	**Weight Norm**
Point **p**	$\lVert\mathbf{p}\rVert_\bullet = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2}$	$\lVert\mathbf{p}\rVert_\circ = \lvert p_w \rvert \mathbb{1}$
Line ***l***	$\lVert\boldsymbol{l}\rVert_\bullet = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$	$\lVert\boldsymbol{l}\rVert_\circ = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$
Plane **g**	$\lVert\mathbf{g}\rVert_\bullet = \lvert g_w \rvert \mathbf{1}$	$\lVert\mathbf{g}\rVert_\circ = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$

Unitization

- An object is *unitized* when its weight has magnitude one

Type	**Definition**	**Unitization**
Point $\mathbf{p}$	$\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4$	$p_w^2 = 1$
Line $\boldsymbol{l}$	$\boldsymbol{l} = l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43} + l_{mx} \mathbf{e}_{23} + l_{my} \mathbf{e}_{31} + l_{mz} \mathbf{e}_{12}$	$l_{vx}^2 + l_{vy}^2 + l_{vz}^2 = 1$
Plane $\mathbf{g}$	$\mathbf{g} = g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} + g_w \mathbf{e}_{321}$	$g_x^2 + g_y^2 + g_z^2 = 1$

Geometric Norm

- Bulk and weight norms by themselves not meaningful
- But add them, and result is a *homogeneous magnitude*
- Represents distance from origin
- Called the geometric norm

$$\|\mathbf{u}\| = \|\mathbf{u}\|_{\bullet} + \|\mathbf{u}\|_{\circ} = \sqrt{\mathbf{u} \bullet \mathbf{u}} + \sqrt{\mathbf{u} \circ \mathbf{u}}$$

- Two-component quantity, sum of scalar and antiscalar
- Can be unitized by making weight one

Sec. 2.10.3

Geometric Norm

Type	**Geometric Norm**	**Interpretation**
Point **p**	$\widehat{\lVert \mathbf{p} \rVert} = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from the origin to the point **p**.
Line ***l***	$\widehat{\lVert \boldsymbol{l} \rVert} = \dfrac{\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}}{\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$	Perpendicular distance from the origin to the line ***l***.
Plane **g**	$\widehat{\lVert \mathbf{g} \rVert} = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$	Perpendicular distance from the origin to the plane **g**.

Attitude

- Weight components contain attitude information
- Attitude can be extracted as directed length / area

$$\operatorname{att}(\mathbf{u}) = \mathbf{u} \vee \overline{\mathbf{e}}_4$$

Type	**Attitude**
Point $\mathbf{p}$	$\operatorname{att}(\mathbf{p}) = p_w \mathbf{1}$
Line $\boldsymbol{l}$	$\operatorname{att}(\boldsymbol{l}) = l_{vx}\mathbf{e}_1 + l_{vy}\mathbf{e}_2 + l_{vz}\mathbf{e}_3$
Plane $\mathbf{g}$	$\operatorname{att}(\mathbf{g}) = g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31} + g_z\mathbf{e}_{12}$

Euclidean Distance

- Weight of a product contains information about volume
- But it also includes weights of objects multiplied together
- Euclidean distance given by quotient

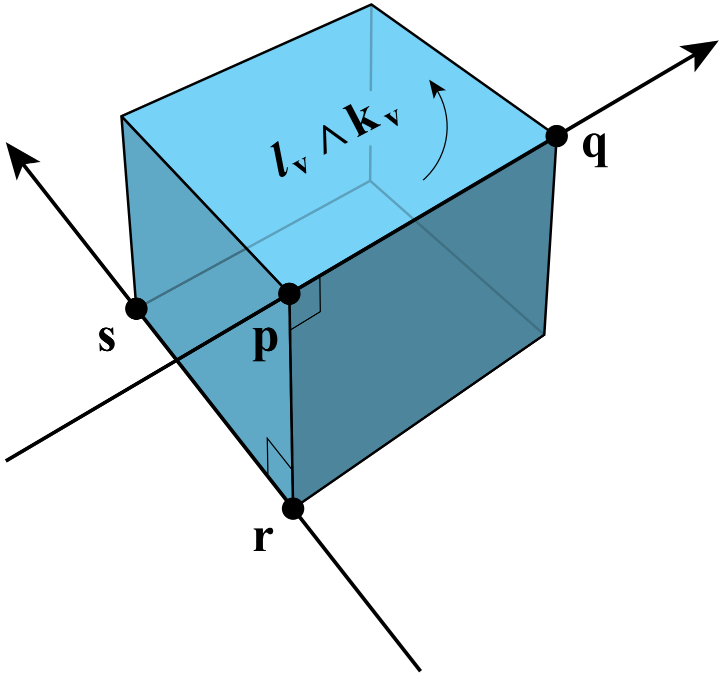
$$d\left(\mathbf{a},\mathbf{b}\right)=\frac{\left\|\operatorname{att}\left(\mathbf{a}\wedge\mathbf{b}\right)\right\|_{\bullet}}{\left\|\operatorname{att}\left(\mathbf{a}\right)\wedge\operatorname{att}\left(\mathbf{b}\right)\right\|_{\bullet}}$$

- Result is a volume divided by an area

Euclidean Distance

$\boldsymbol{l} = \mathbf{p} \wedge \mathbf{q}$

$\mathbf{k} = \mathbf{r} \wedge \mathbf{s}$



$\boldsymbol{l}_\mathrm{v} = \mathbf{q} - \mathbf{p}$

$\mathbf{k}_\mathrm{v} = \mathbf{s} - \mathbf{r}$

Euclidean Distance

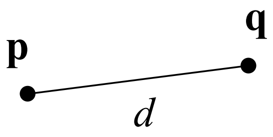
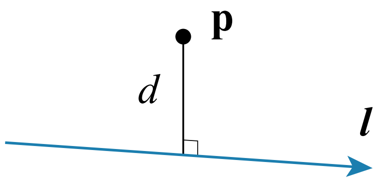
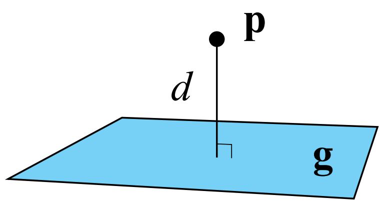
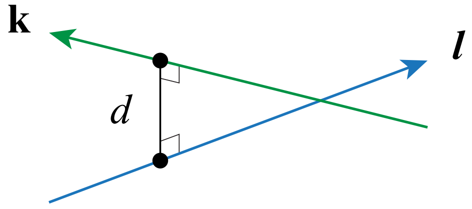
- Distance formula can be transformed into homogeneous magnitude:

$$d\left(\mathbf{a},\mathbf{b}\right)=\left\|\operatorname{att}\left(\mathbf{a}\wedge\mathbf{b}\right)\right\|_{\bullet}+\left\|\mathbf{a}\wedge\operatorname{att}\left(\mathbf{b}\right)\right\|_{\circ}$$

- Sometimes, a signed distance is meaningful:

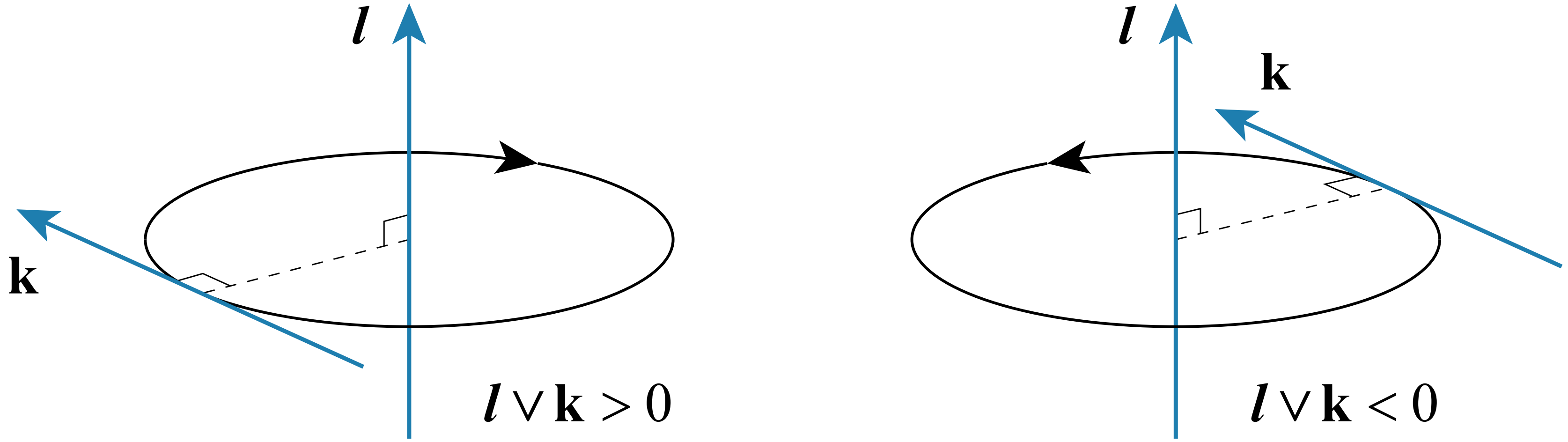
$$d\left(\mathbf{a},\mathbf{b}\right)=\begin{cases}\mathbf{a}\vee\mathbf{b}+\left\|\mathbf{a}\wedge\operatorname{att}\left(\mathbf{b}\right)\right\|_{\circ}, & \text{if } \operatorname{gr}\left(\mathbf{a}\right)+\operatorname{gr}\left(\mathbf{b}\right)=n;\\ \left\|\operatorname{att}\left(\mathbf{a}\wedge\mathbf{b}\right)\right\|_{\bullet}+\left\|\mathbf{a}\wedge\operatorname{att}\left(\mathbf{b}\right)\right\|_{\circ}, & \text{otherwise.}\end{cases}$$

Euclidean Distance

Distance Formula	Illustration
Distance d between points **p** and **q**. $d(\mathbf{p},\mathbf{q}) = \lVert \mathbf{q}_{xyz} p_w - \mathbf{p}_{xyz} q_w \rVert \mathbf{1} + \lvert p_w q_w \rvert \mathbb{1}$	
Perpendicular distance d between point **p** and line ***l***. $d(\mathbf{p},\boldsymbol{l}) = \lVert \boldsymbol{l}_{\mathbf{v}} \times \mathbf{p}_{xyz} + p_w \boldsymbol{l}_{\mathbf{m}} \rVert \mathbf{1} + \lVert p_w \boldsymbol{l}_{\mathbf{v}} \rVert \mathbb{1}$	
Perpendicular distance d between point **p** and plane **g**. $d(\mathbf{p},\mathbf{g}) = (\mathbf{p} \cdot \mathbf{g}) \mathbf{1} + \lVert p_w \mathbf{g}_{xyz} \rVert \mathbb{1}$	
Perpendicular distance d between skew lines ***l*** and **k**. $d(\boldsymbol{l},\mathbf{k}) = -(\boldsymbol{l}_{\mathbf{v}} \cdot \mathbf{k}_{\mathbf{m}} + \boldsymbol{l}_{\mathbf{m}} \cdot \mathbf{k}_{\mathbf{v}}) \mathbf{1} + \lVert \boldsymbol{l}_{\mathbf{v}} \times \mathbf{k}_{\mathbf{v}} \rVert \mathbb{1}$	

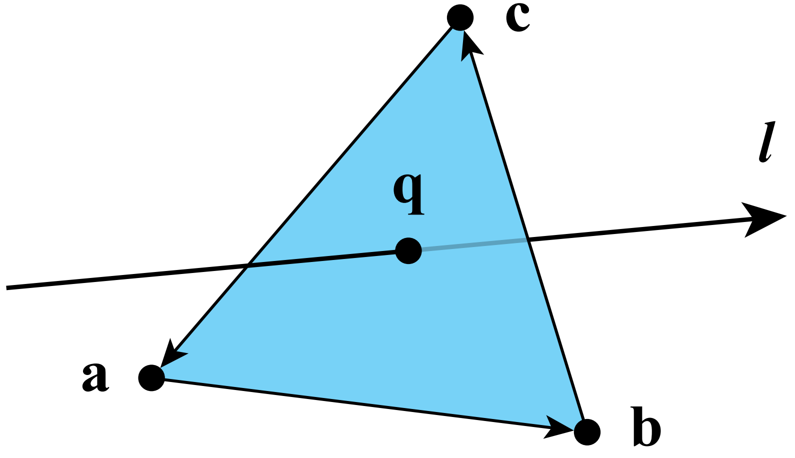
Line Crossing

- Sign of wedge product between lines gives crossing orientation



Line-Triangle Intersection

- Wedge product with all three edges of CCW-wound triangle must be positive



Bulk and Weight Duals

- Multiply by metric or antimetric, then take complement
- Bulk dual: $\mathbf{u}^{\bigstar} = \overline{\mathbf{G}\mathbf{u}}$ $\mathbf{u}_{\bigstar} = \underline{\mathbf{G}\mathbf{u}}$
- Weight dual: $\mathbf{u}^{\medwhitestar} = \overline{\mathbb{G}\mathbf{u}}$ $\mathbf{u}_{\medwhitestar} = \underline{\mathbb{G}\mathbf{u}}$

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{u}^{\bigstar}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0
$\mathbf{u}_{\bigstar}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$\mathbf{e}_4$	0
$\mathbf{u}^{\medwhitestar}$	0	0	0	0	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	0	$\mathbf{1}$
$\mathbf{u}_{\medwhitestar}$	0	0	0	0	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	$\mathbf{1}$

Interior Products

- Two exterior products combined with two duals
- Eight *interior* products using right and left duals

• Bulk contraction	$\mathbf{a} \vee \mathbf{b}^{\text{★}}$	$\mathbf{b}_{\text{★}} \vee \mathbf{a}$
• Weight contraction	$\mathbf{a} \vee \mathbf{b}^{\text{☆}}$	$\mathbf{b}_{\text{☆}} \vee \mathbf{a}$
• Bulk expansion	$\mathbf{a} \wedge \mathbf{b}^{\text{★}}$	$\mathbf{b}_{\text{★}} \wedge \mathbf{a}$
• Weight expansion	$\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$	$\mathbf{b}_{\text{☆}} \wedge \mathbf{a}$

Interior Products

- Right and left interior products differ by grade-dependent sign:

$$\mathbf{b}_{\ast} \vee \mathbf{a} = (-1)^{\mathrm{gr}(\mathbf{b})[\mathrm{gr}(\mathbf{a})+\mathrm{gr}(\mathbf{b})]}\, \mathbf{a} \vee \mathbf{b}^{\ast}$$

$$\mathbf{b}_{\ast} \wedge \mathbf{a} = (-1)^{\mathrm{ag}(\mathbf{b})[\mathrm{ag}(\mathbf{a})+\mathrm{ag}(\mathbf{b})]}\, \mathbf{a} \wedge \mathbf{b}^{\ast}$$

- Here, $\ast$ is either ★ or ☆

- Really need only four interior products

Interior Products

- Interior products reduce to inner products for same grade:

$$\mathbf{a} \vee \mathbf{b}^{\bigstar} = \mathbf{a} \bullet \mathbf{b}, \qquad \text{when } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b})$$

$$\mathbf{a} \vee \mathbf{b}^{\text{☆}} = (\mathbf{a} \circ \mathbf{b}) \vee \mathbf{1}, \qquad \text{when } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b})$$

$$\mathbf{a} \wedge \mathbf{b}^{\text{☆}} = \mathbf{a} \circ \mathbf{b}, \qquad \text{when } \operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b})$$

$$\mathbf{a} \wedge \mathbf{b}^{\bigstar} = (\mathbf{a} \bullet \mathbf{b}) \wedge \mathbb{1}, \qquad \text{when } \operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b})$$

Euclidean Angle

- Canonical angle given by dot product:

$$\cos\phi = \frac{\mathbf{a}\cdot\mathbf{b}}{\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|}$$

- This generalizes with bulk contraction:

$$\cos\phi = \frac{\left\|\mathbf{a}\vee\mathbf{b}^{\bigstar}\right\|_{\bullet}}{\left\|\mathbf{a}\right\|_{\bullet}\left\|\mathbf{b}\right\|_{\bullet}}$$

Euclidean Angle

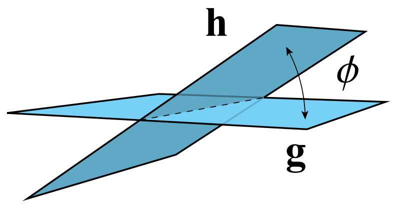
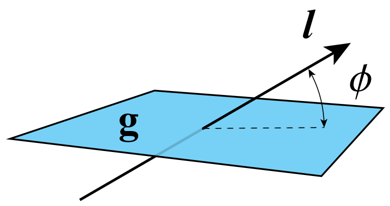
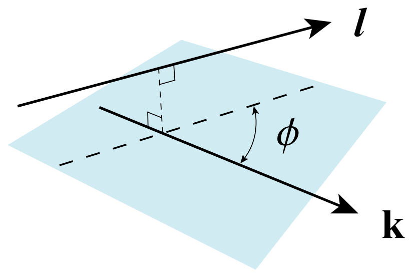
- Formula can be transformed into homogeneous magnitude:

$$\cos\phi = \left\| \mathbf{a} \vee \mathbf{b}^{\star} \right\|_{\bullet} + \left\| \mathbf{a} \right\|_{\circ} \left\| \mathbf{b} \right\|_{\circ}$$

- When grades equal, positive and negative angles make sense:

$$\cos\phi(\mathbf{a},\mathbf{b}) = \begin{cases} \mathbf{a} \vee \mathbf{b}^{\star} + \left\| \mathbf{a} \right\|_{\circ} \left\| \mathbf{b} \right\|_{\circ}, & \text{if } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b}); \\ \left\| \mathbf{a} \vee \mathbf{b}^{\star} \right\|_{\bullet} + \left\| \mathbf{a} \right\|_{\circ} \left\| \mathbf{b} \right\|_{\circ}, & \text{otherwise.} \end{cases}$$

Euclidean Angle

Angle Formula	Illustration
Cosine of angle ϕ between planes **g** and **h**. $\cos\phi(\mathbf{g},\mathbf{h}) = (\mathbf{g}_{xyz} \cdot \mathbf{h}_{xyz})\mathbf{1} + \lVert\mathbf{g}\rVert_{\circ}\lVert\mathbf{h}\rVert_{\circ}$	
Cosine of angle ϕ between plane **g** and line ***l***. $\cos\phi(\mathbf{g},\boldsymbol{l}) = \lVert\mathbf{g}_{xyz} \times \boldsymbol{l}_{\mathbf{v}}\rVert\mathbf{1} + \lVert\mathbf{g}\rVert_{\circ}\lVert\boldsymbol{l}\rVert_{\circ}$	
Cosine of angle ϕ between lines ***l*** and **k**. $\cos\phi(\boldsymbol{l},\mathbf{k}) = (\boldsymbol{l}_{\mathbf{v}} \cdot \mathbf{k}_{\mathbf{v}})\mathbf{1} + \lVert\boldsymbol{l}\rVert_{\circ}\lVert\mathbf{k}\rVert_{\circ}$	

Parametric Formulas

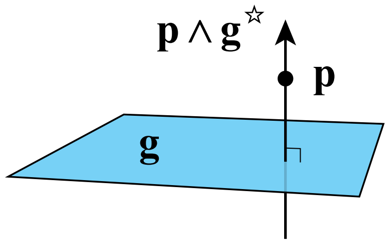
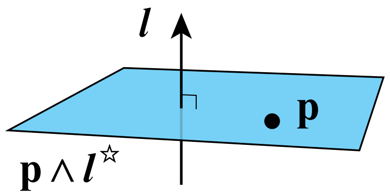
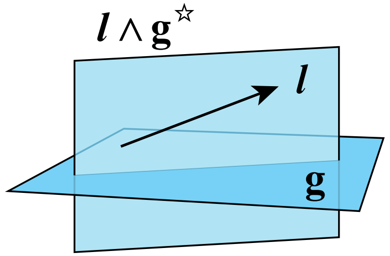
- Line or plane **u** can be expressed parametrically:

$$\mathbf{p}(\alpha) = \mathbf{p}_0 + \operatorname{att}(\mathbf{u}) \vee \alpha^{\star}$$

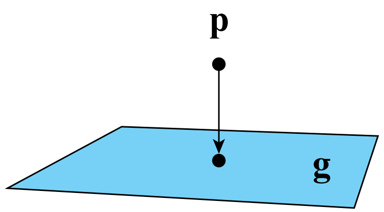
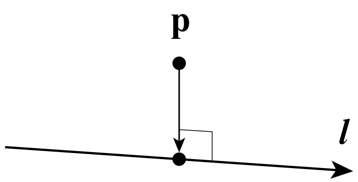
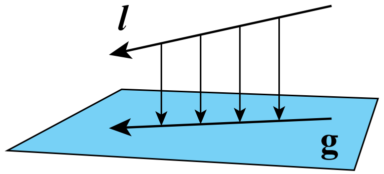
- α is an arbitrary parameter having grade two less than **u**

- This formula surprisingly holds in conformal algebras as well

Weight Expansion

Expansion Operation	**Illustration**
Line containing point **p** and orthogonal to plane **g**. $\mathbf{p} \wedge \mathbf{g}^{\star} = -p_w g_x \mathbf{e}_{41} - p_w g_y \mathbf{e}_{42} - p_w g_z \mathbf{e}_{43}$ $+ (p_z g_y - p_y g_z)\mathbf{e}_{23} + (p_x g_z - p_z g_x)\mathbf{e}_{31} + (p_y g_x - p_x g_y)\mathbf{e}_{12}$	
Plane containing point **p** and orthogonal to line ***l***. $\mathbf{p} \wedge \boldsymbol{l}^{\star} = -p_w l_{vx} \mathbf{e}_{423} - p_w l_{vy} \mathbf{e}_{431} - p_w l_{vz} \mathbf{e}_{412}$ $+ (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\mathbf{e}_{321}$	
Plane containing line ***l*** and orthogonal to plane **g**. $\boldsymbol{l} \wedge \mathbf{g}^{\star} = (l_{vy} g_z - l_{vz} g_y)\mathbf{e}_{423} + (l_{vz} g_x - l_{vx} g_z)\mathbf{e}_{431} + (l_{vx} g_y - l_{vy} g_x)\mathbf{e}_{412}$ $-(l_{mx} g_x + l_{my} g_y + l_{mz} g_z)\mathbf{e}_{321}$	

Orthogonal Projection

Projection Operation	**Illustration**
Orthogonal projection of point **p** onto plane **g**. $\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{\star}) = (g_x^2 + g_y^2 + g_z^2)(p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4)$ $- (g_x p_x + g_y p_y + g_z p_z + g_w p_w)(g_x\mathbf{e}_1 + g_y\mathbf{e}_2 + g_z\mathbf{e}_3)$	
Orthogonal projection of point **p** onto line ***l***. $\boldsymbol{l} \vee (\mathbf{p} \wedge \boldsymbol{l}^{\star}) = (l_{vx}p_x + l_{vy}p_y + l_{vz}p_z)(l_{vx}\mathbf{e}_1 + l_{vy}\mathbf{e}_2 + l_{vz}\mathbf{e}_3) + (l_{vx}^2 + l_{vy}^2 + l_{vz}^2)\,p_w\mathbf{e}_4$ $+ (l_{vy}l_{mz} - l_{vz}l_{my})\,p_w\mathbf{e}_1 + (l_{vz}l_{mx} - l_{vx}l_{mz})\,p_w\mathbf{e}_2 + (l_{vx}l_{my} - l_{vy}l_{mx})\,p_w\mathbf{e}_3$	
Orthogonal projection of line ***l*** onto plane **g**. $\mathbf{g} \vee (\boldsymbol{l} \wedge \mathbf{g}^{\star}) = (g_x^2 + g_y^2 + g_z^2)(l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43})$ $- (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})(g_x\mathbf{e}_{41} + g_y\mathbf{e}_{42} + g_z\mathbf{e}_{43})$ $+ (g_x l_{mx} + g_y l_{my} + g_z l_{mz})(g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31} + g_z\mathbf{e}_{12})$ $+ (g_z l_{vy} - g_y l_{vz})\,g_w\mathbf{e}_{23} + (g_x l_{vz} - g_z l_{vx})\,g_w\mathbf{e}_{31} + (g_y l_{vx} - g_x l_{vy})\,g_w\mathbf{e}_{12}$	

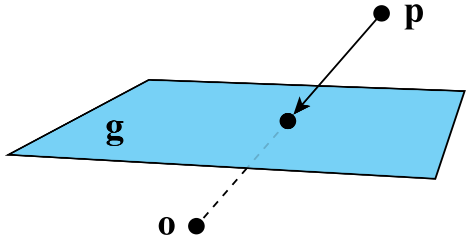
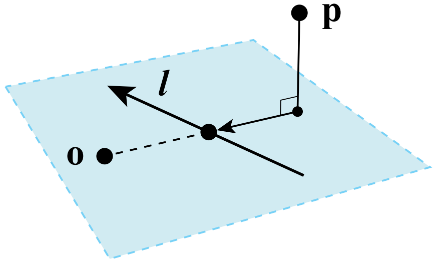
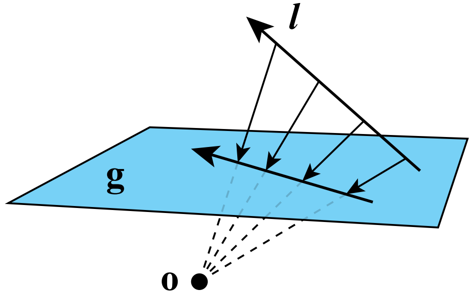
Support

- Orthogonal projection of origin onto line or plane
- Support is point closest to origin contained by object

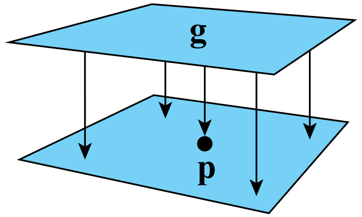
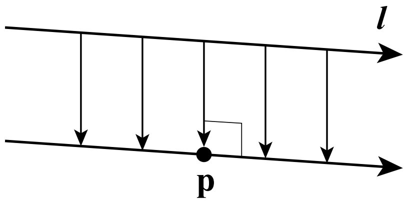
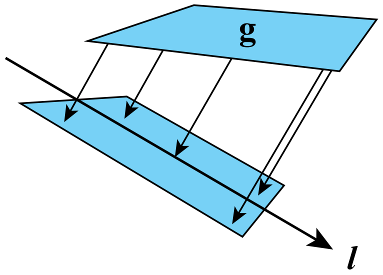
$$\operatorname{sup}(\boldsymbol{l}) = (l_{vy}l_{mz} - l_{vz}l_{my})\,\mathbf{e}_1 + (l_{vz}l_{mx} - l_{vx}l_{mz})\,\mathbf{e}_2 + (l_{vx}l_{my} - l_{vy}l_{mx})\,\mathbf{e}_3 + \boldsymbol{l}_{\mathbf{v}}^2\mathbf{e}_4$$

$$\operatorname{sup}(\mathbf{g}) = -g_x g_w \mathbf{e}_1 - g_y g_w \mathbf{e}_2 - g_z g_w \mathbf{e}_3 + \left(g_x^2 + g_y^2 + g_z^2\right)\mathbf{e}_4$$

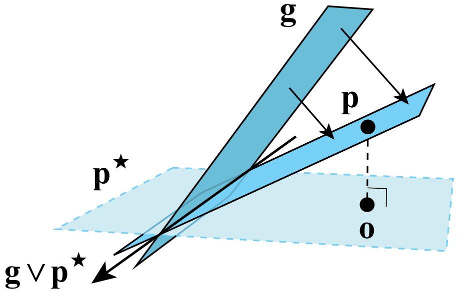
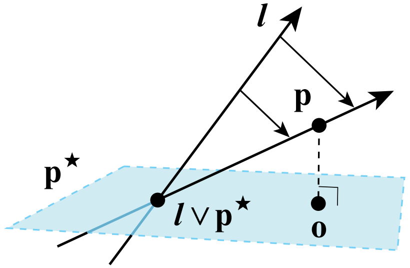
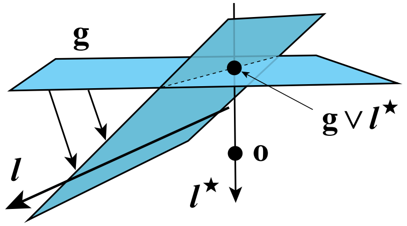
Central Projection

Projection Operation	**Illustration**
Central projection of point **p** onto plane **g**. $\mathbf{g} \vee \left(\mathbf{p} \wedge \mathbf{g}^{\star}\right) = g_w^2\,(p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3)$ $-(g_x p_x + g_y p_y + g_z p_z)\,g_w\mathbf{e}_4$	
Central projection of point **p** onto line ***l***. $\boldsymbol{l} \vee \left(\mathbf{p} \wedge \boldsymbol{l}^{\star}\right) = \left(l_{mx}^2 + l_{my}^2 + l_{mz}^2\right)(p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3)$ $-(l_{mx}p_x + l_{my}p_y + l_{mz}p_z)(l_{mx}\mathbf{e}_1 + l_{my}\mathbf{e}_2 + l_{mz}\mathbf{e}_3)$ $+(l_{mx}\,(l_{vz}p_y - l_{vy}p_z) + l_{my}\,(l_{vx}p_z - l_{vz}p_x) + l_{mz}\,(l_{vy}p_x - l_{vx}p_y))\,\mathbf{e}_4$	
Central projection of line ***l*** onto plane **g**. $\mathbf{g} \vee \left(\boldsymbol{l} \wedge \mathbf{g}^{\star}\right) = (g_y l_{mz} - g_z l_{my})\,g_w\mathbf{e}_{41} + g_w^2 l_{mx}\mathbf{e}_{23}$ $+(g_z l_{mx} - g_x l_{mz})\,g_w\mathbf{e}_{42} + g_w^2 l_{my}\mathbf{e}_{31}$ $+(g_x l_{my} - g_y l_{mx})\,g_w\mathbf{e}_{43} + g_w^2 l_{mz}\mathbf{e}_{12}$	

Orthogonal Antiprojection

Projection Operation	**Illustration**
Orthogonal antiprojection of plane **g** onto point **p**. $\mathbf{p} \wedge \left(\mathbf{g} \vee \mathbf{p}^{\star} \right) = p_w^2 \left(g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} \right) - \left(p_x g_x + p_y g_y + p_z g_z \right) p_w \mathbf{e}_{321}$	
Orthogonal antiprojection of line ***l*** onto point **p**. $\mathbf{p} \wedge \left(\boldsymbol{l} \vee \mathbf{p}^{\star} \right) = p_w^2 \left(l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43} \right) + \left(p_y l_{vz} - p_z l_{vy} \right) p_w \mathbf{e}_{23} + \left(p_z l_{vx} - p_x l_{vz} \right) p_w \mathbf{e}_{31} + \left(p_x l_{vy} - p_y l_{vx} \right) p_w \mathbf{e}_{12}$	
Orthogonal antiprojection of plane **g** onto line ***l***. $\boldsymbol{l} \wedge \left(\mathbf{g} \vee \boldsymbol{l}^{\star} \right) = \left(l_{vx}^2 + l_{vy}^2 + l_{vz}^2 \right) \left(g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} \right) - \left(l_{vx} g_x + l_{vy} g_y + l_{vz} g_z \right) \left(l_{vx} \mathbf{e}_{423} + l_{vy} \mathbf{e}_{431} + l_{vz} \mathbf{e}_{412} \right) + \left(l_{vz} l_{my} - l_{vy} l_{mz} \right) g_x \mathbf{e}_{321} + \left(l_{vx} l_{mz} - l_{vz} l_{mx} \right) g_y \mathbf{e}_{321} + \left(l_{vy} l_{mx} - l_{vx} l_{my} \right) g_z \mathbf{e}_{321}$	

Central Antiprojection

Projection Operation	**Illustration**
Central antiprojection of plane $\mathbf{g}$ onto point $\mathbf{p}$. $\mathbf{p} \wedge (\mathbf{g} \vee \mathbf{p}^\star) = \left[(p_y^2 + p_z^2)\, g_x - (p_y g_y + p_z g_z + p_w g_w)\, p_x \right] \mathbf{e}_{423}$ $+ \left[(p_z^2 + p_x^2)\, g_y - (p_x g_x + p_z g_z + p_w g_w)\, p_y \right] \mathbf{e}_{431}$ $+ \left[(p_x^2 + p_y^2)\, g_z - (p_x g_x + p_y g_y + p_w g_w)\, p_z \right] \mathbf{e}_{412}$ $+ (p_x^2 + p_y^2 + p_z^2)\, g_w \mathbf{e}_{321}$	
Central antiprojection of line $\boldsymbol{l}$ onto point $\mathbf{p}$. $\mathbf{p} \wedge (\boldsymbol{l} \vee \mathbf{p}^\star) = (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})(p_x \mathbf{e}_{41} + p_y \mathbf{e}_{42} + p_z \mathbf{e}_{43})$ $+ (p_y^2 + p_z^2)\, l_{mx} \mathbf{e}_{23} + (p_z^2 + p_x^2)\, l_{my} \mathbf{e}_{31} + (p_x^2 + p_y^2)\, l_{mz} \mathbf{e}_{12}$ $+ (p_z l_{my} - p_y l_{mz})\, p_w \mathbf{e}_{41} - (p_y l_{my} + p_z l_{mz})\, p_x \mathbf{e}_{23}$ $+ (p_x l_{mz} - p_z l_{mx})\, p_w \mathbf{e}_{42} - (p_z l_{mz} + p_x l_{mx})\, p_y \mathbf{e}_{31}$ $+ (p_y l_{mx} - p_x l_{my})\, p_w \mathbf{e}_{43} - (p_x l_{mx} + p_y l_{my})\, p_z \mathbf{e}_{12}$	
Central antiprojection of plane $\mathbf{g}$ onto line $\boldsymbol{l}$. $\boldsymbol{l} \wedge (\mathbf{g} \vee \boldsymbol{l}^\star) = (l_{mx} g_x + l_{my} g_y + l_{mz} g_z)(l_{mx} \mathbf{e}_{423} + l_{my} \mathbf{e}_{431} + l_{mz} \mathbf{e}_{412})$ $+ (l_{my} l_{vz} - l_{mz} l_{vy})\, g_w \mathbf{e}_{423} + (l_{mz} l_{vx} - l_{mx} l_{vz})\, g_w \mathbf{e}_{431}$ $+ (l_{mx} l_{vy} - l_{my} l_{vx})\, g_w \mathbf{e}_{412} + (l_{mx}^2 + l_{my}^2 + l_{mz}^2)\, g_w \mathbf{e}_{321}$	

Antisupport

- Central antiprojection of horizon onto point or line
- Antisupport is plane farthest from origin containing object

$$\operatorname{asp}(\mathbf{p}) = -p_x p_w \mathbf{e}_{423} - p_y p_w \mathbf{e}_{431} - p_z p_w \mathbf{e}_{412} + \left(p_x^2 + p_y^2 + p_z^2\right)\mathbf{e}_{321}$$

$$\operatorname{asp}(\boldsymbol{l}) = (l_{vz} l_{my} - l_{vy} l_{mz})\mathbf{e}_{423} + (l_{vx} l_{mz} - l_{vz} l_{mx})\mathbf{e}_{431} + (l_{vy} l_{mx} - l_{vx} l_{my})\mathbf{e}_{412} + \boldsymbol{l}_{\mathbf{m}}^2 \mathbf{e}_{321}$$

Conformal Exterior Algebra

- 5D algebra modeling 3D geometry and motion

$$\mathfrak{g}_{\pm} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$

$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_- \cdot \mathbf{e}_- = -1$$

$$\mathbf{e}_+ \cdot \mathbf{e}_+ = +1$$

Conformal Exterior Algebra

- It is convenient to change the basis as follows

$$\mathbf{e}_4 = \tfrac{1}{2}\left(\mathbf{e}_- - \mathbf{e}_+\right)$$

$$\mathbf{e}_5 = \mathbf{e}_- + \mathbf{e}_+$$

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$

$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_4 \cdot \mathbf{e}_5 = -1$$

Conformal Basis Elements

Type	**Grade**	**Basis Elements**
Scalar	0	$\mathbf{1}$
Vectors	1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5$
Bivectors	2	$\mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{15}, \mathbf{e}_{25}, \mathbf{e}_{35}, \mathbf{e}_{45}$
Trivectors	3	$\mathbf{e}_{423}, \mathbf{e}_{431}, \mathbf{e}_{412}, \mathbf{e}_{321}, \mathbf{e}_{415}, \mathbf{e}_{425}, \mathbf{e}_{435}, \mathbf{e}_{235}, \mathbf{e}_{315}, \mathbf{e}_{125}$
Quadrivectors	4	$\mathbf{e}_{1234}, \mathbf{e}_{4235}, \mathbf{e}_{4315}, \mathbf{e}_{4125}, \mathbf{e}_{3215}$
Antiscalar	5	$\mathbb{1} = \mathbf{e}_{12345}$

Special Points

- $\mathbf{e}_4$ still represents the origin

- $\mathbf{e}_5$ represents the point at infinity in a stereographic projection

Flat Objects

- Everything from PGA appears in CGA with factor of $\mathbf{e}_5$

$$\mathbf{p} = p_x \mathbf{e}_{15} + p_y \mathbf{e}_{25} + p_z \mathbf{e}_{35} + p_w \mathbf{e}_{45}$$

$$\boldsymbol{l} = l_{vx} \mathbf{e}_{415} + l_{vy} \mathbf{e}_{425} + l_{vz} \mathbf{e}_{435} + l_{mx} \mathbf{e}_{235} + l_{my} \mathbf{e}_{315} + l_{mz} \mathbf{e}_{125}$$

$$\mathbf{g} = g_x \mathbf{e}_{4235} + g_y \mathbf{e}_{4315} + g_z \mathbf{e}_{4125} + g_w \mathbf{e}_{3215}$$

Round Objects

- We also have four new types of round object
  - Round points
  - Dipoles
  - Circles
  - Spheres
- Flat points, lines, and planes are special cases of dipoles, circles, and spheres that include the point at infinity

Round Point

$$\mathbf{a} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + \mathbf{e}_4 + \frac{\mathbf{p}^2 + r^2}{2}\mathbf{e}_5$$

$$\mathbf{a} = \underbrace{a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3 + a_w\mathbf{e}_4}_{\text{Carrier Point}} + \underbrace{a_u\mathbf{e}_5}_{\text{Infinity}},$$

$\left(\text{when } a_x = a_y = a_z = a_w = 0\right)$

Dipole

$$\mathbf{d} = n_x \mathbf{e}_{41} + n_y \mathbf{e}_{42} + n_z \mathbf{e}_{43} + \left(p_y n_z - p_z n_y \right) \mathbf{e}_{23} + \left(p_z n_x - p_x n_z \right) \mathbf{e}_{31} + \left(p_x n_y - p_y n_x \right) \mathbf{e}_{12}$$

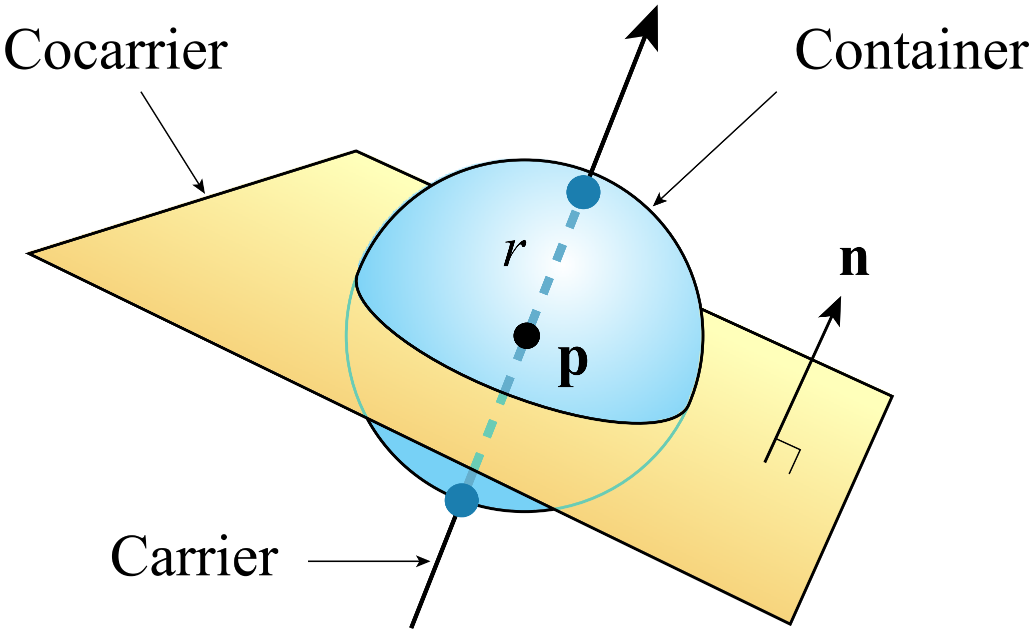
$$+ (\mathbf{p} \cdot \mathbf{n}) \left(p_x \mathbf{e}_{15} + p_y \mathbf{e}_{25} + p_z \mathbf{e}_{35} + \mathbf{e}_{45} \right) - \frac{\mathbf{p}^2 + r^2}{2} \left(n_x \mathbf{e}_{15} + n_y \mathbf{e}_{25} + n_z \mathbf{e}_{35} \right)$$

$$\mathbf{d} = \underbrace{\overbrace{d_{vx} \mathbf{e}_{41} + d_{vy} \mathbf{e}_{42} + d_{vz} \mathbf{e}_{43}}^{\text{Cocarrier Normal}} + d_{mx} \mathbf{e}_{23} + d_{my} \mathbf{e}_{31} + d_{mz} \mathbf{e}_{12}}_{\text{Carrier Line}} + \underbrace{d_{px} \mathbf{e}_{15} + d_{py} \mathbf{e}_{25} + d_{pz} \mathbf{e}_{35} + \overbrace{d_{pw} \mathbf{e}_{45}}^{\text{Cocarrier Position}}}_{\text{Flat Point}}.$$

$\left(\text{when } d_{vz} = d_{vy} = d_{vz} = d_{mx} = d_{my} = d_{mz} = 0\right)$

Dipole

- A dipole is a one-dimensional sphere



Circle

$$\mathbf{c} = n_x\mathbf{e}_{423} + n_y\mathbf{e}_{431} + n_z\mathbf{e}_{412} + (p_y n_z - p_z n_y)\mathbf{e}_{415} + (p_z n_x - p_x n_z)\mathbf{e}_{425} + (p_x n_y - p_y n_x)\mathbf{e}_{435}$$

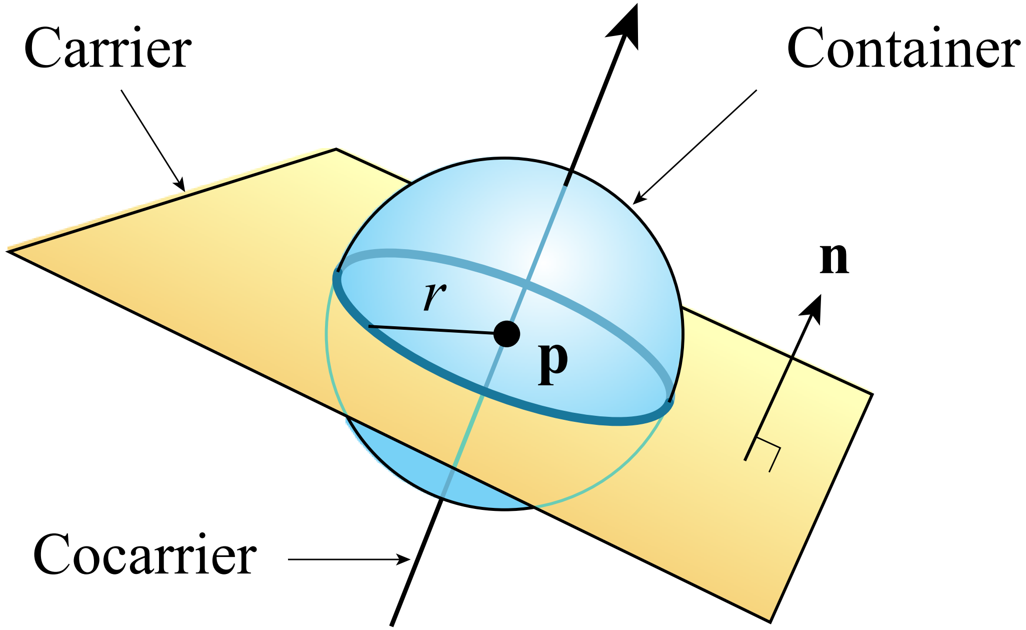
$$+(\mathbf{p}\cdot\mathbf{n})(p_x\mathbf{e}_{235} + p_y\mathbf{e}_{315} + p_z\mathbf{e}_{125} - \mathbf{e}_{321}) - \frac{\mathbf{p}^2 - r^2}{2}(n_x\mathbf{e}_{235} + n_y\mathbf{e}_{315} + n_z\mathbf{e}_{125})$$

$$\mathbf{c} = \underbrace{\overbrace{c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412}}^{\text{Cocarrier Direction}} + c_{gw}\mathbf{e}_{321}}_{\text{Carrier Plane}} + \underbrace{\overbrace{c_{vx}\mathbf{e}_{415} + c_{vy}\mathbf{e}_{425} + c_{vz}\mathbf{e}_{435}}^{\text{Cocarrier Moment}} + c_{mx}\mathbf{e}_{235} + c_{my}\mathbf{e}_{315} + c_{mz}\mathbf{e}_{125}}_{\text{Flat Line}}.$$

$(\text{when } \mathrm{c}_{gx} = c_{gy} = c_{gz} = c_{gw} = 0)$

Circle

- A circle is a two-dimensional sphere



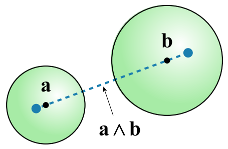
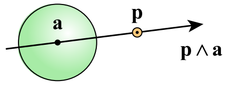
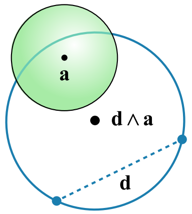
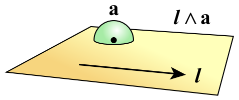
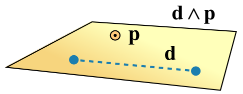
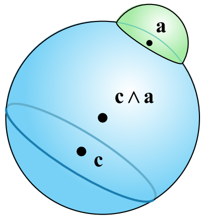
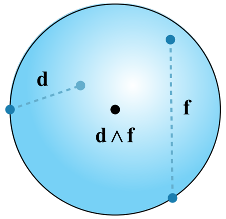
Sphere

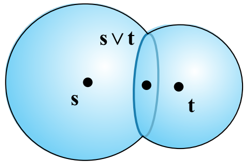
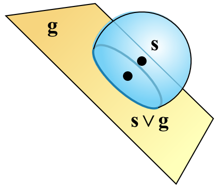
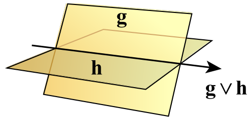
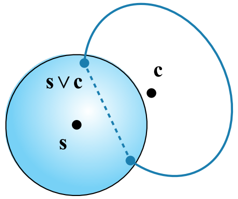
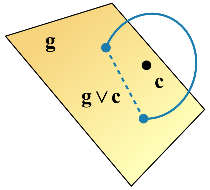
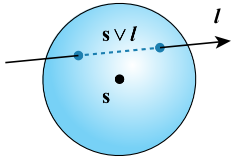
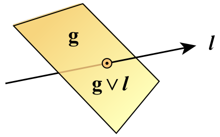
$$\mathbf{s} = p_x \mathbf{e}_{4235} + p_y \mathbf{e}_{4315} + p_z \mathbf{e}_{4125} - \mathbf{e}_{1234} - \frac{\mathbf{p}^2 - r^2}{2} \mathbf{e}_{3215}$$

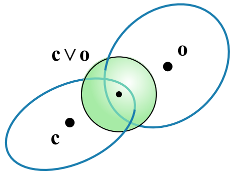
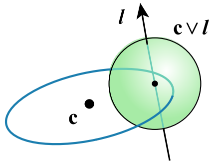
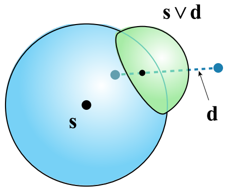
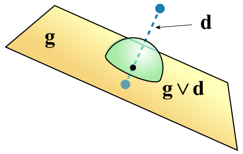
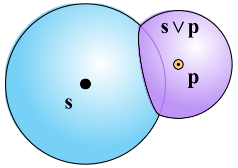
$$\mathbf{s} = \underbrace{s_u \mathbf{e}_{1234}}_{\text{Carrier Space}} + \underbrace{s_x \mathbf{e}_{4235} + s_y \mathbf{e}_{4315} + s_z \mathbf{e}_{4125} + s_w \mathbf{e}_{3215}}_{\substack{\text{Flat Plane} \\ (\text{when } s_u = 0)}},$$

Join and Meet

- Objects joined with wedge product
- Intersection calculated with antiwedge product
- Same math as PGA

Join Operation	Illustration
Dipole containing round points **a** and **b**. $\mathbf{a} \wedge \mathbf{b} = (a_w b_x - a_x b_w)\,\mathbf{e}_{41} + (a_w b_y - a_y b_w)\,\mathbf{e}_{42} + (a_w b_z - a_z b_w)\,\mathbf{e}_{43}$ $+ (a_y b_z - a_z b_y)\,\mathbf{e}_{23} + (a_z b_x - a_x b_z)\,\mathbf{e}_{31} + (a_x b_y - a_y b_x)\,\mathbf{e}_{12}$ $+ (a_x b_u - a_u b_x)\,\mathbf{e}_{15} + (a_y b_u - a_u b_y)\,\mathbf{e}_{25}$ $+ (a_z b_u - a_u b_z)\,\mathbf{e}_{35} + (a_w b_u - a_u b_w)\,\mathbf{e}_{45}$	
Line containing flat point **p** and round point **a**. $\mathbf{p} \wedge \mathbf{a} = (p_x a_w - p_w a_x)\,\mathbf{e}_{415} + (p_z a_y - p_y a_z)\,\mathbf{e}_{235}$ $+ (p_y a_w - p_w a_y)\,\mathbf{e}_{425} + (p_x a_z - p_z a_x)\,\mathbf{e}_{315}$ $+ (p_z a_w - p_w a_z)\,\mathbf{e}_{435} + (p_y a_x - p_x a_y)\,\mathbf{e}_{125}$	
Circle containing dipole **d** and round point **a**. $\mathbf{d} \wedge \mathbf{a} = (d_{vy} a_z - d_{vz} a_y + d_{mx} a_w)\,\mathbf{e}_{423} + (d_{vz} a_x - d_{vx} a_z + d_{my} a_w)\,\mathbf{e}_{431}$ $+ (d_{vx} a_y - d_{vy} a_x + d_{mz} a_w)\,\mathbf{e}_{412} - (d_{mx} a_x + d_{my} a_y + d_{mz} a_z)\,\mathbf{e}_{321}$ $+ (d_{px} a_w - d_{pw} a_x + d_{vx} a_u)\,\mathbf{e}_{415} + (d_{pz} a_y - d_{py} a_z + d_{mx} a_u)\,\mathbf{e}_{235}$ $+ (d_{py} a_w - d_{pw} a_y + d_{vy} a_u)\,\mathbf{e}_{425} + (d_{px} a_z - d_{pz} a_x + d_{my} a_u)\,\mathbf{e}_{315}$ $+ (d_{pz} a_w - d_{pw} a_z + d_{vz} a_u)\,\mathbf{e}_{435} + (d_{py} a_x - d_{px} a_y + d_{mz} a_u)\,\mathbf{e}_{125}$	
Plane containing line ***l*** and round point **a**. $\boldsymbol{l} \wedge \mathbf{a} = (l_{vz} a_y - l_{vy} a_z - l_{mx} a_w)\,\mathbf{e}_{4235} + (l_{vx} a_z - l_{vz} a_x - l_{my} a_w)\,\mathbf{e}_{4315}$ $+ (l_{vy} a_x - l_{vx} a_y - l_{mz} a_w)\,\mathbf{e}_{4125} + (l_{mx} a_x + l_{my} a_y + l_{mz} a_z)\,\mathbf{e}_{3215}$	
Plane containing dipole **d** and flat point **p**. $\mathbf{d} \wedge \mathbf{p} = (d_{vy} p_z - d_{vz} p_y + d_{mx} p_w)\,\mathbf{e}_{4235}$ $+ (d_{vz} p_x - d_{vx} p_z + d_{my} p_w)\,\mathbf{e}_{4315}$ $+ (d_{vx} p_y - d_{vy} p_x + d_{mz} p_w)\,\mathbf{e}_{4125}$ $- (d_{mx} p_x + d_{my} p_y + d_{mz} p_z)\,\mathbf{e}_{3215}$	
Sphere containing circle **c** and round point **a**. $\mathbf{c} \wedge \mathbf{a} = -(c_{gx} a_x + c_{gy} a_y + c_{gz} a_z + c_{gw} a_w)\,\mathbf{e}_{1234}$ $+ (c_{vz} a_y - c_{vy} a_z + c_{gx} a_u - c_{mx} a_w)\,\mathbf{e}_{4235}$ $+ (c_{vx} a_z - c_{vz} a_x + c_{gy} a_u - c_{my} a_w)\,\mathbf{e}_{4315}$ $+ (c_{vy} a_x - c_{vx} a_y + c_{gz} a_u - c_{mz} a_w)\,\mathbf{e}_{4125}$ $+ (c_{mx} a_x + c_{my} a_y + c_{mz} a_z + c_{gw} a_u)\,\mathbf{e}_{3215}$	
Sphere containing dipoles **d** and **f**. $\mathbf{d} \wedge \mathbf{f} = -(d_{vx} f_{mx} + d_{vy} f_{my} + d_{vz} f_{mz} + d_{mx} f_{vx} + d_{my} f_{vy} + d_{mz} f_{vz})\,\mathbf{e}_{1234}$ $+ (d_{vy} f_{pz} - d_{vz} f_{py} + d_{pz} f_{vy} - d_{py} f_{vz} + d_{mx} f_{pw} + d_{pw} f_{mx})\,\mathbf{e}_{4235}$ $+ (d_{vz} f_{px} - d_{vx} f_{pz} + d_{px} f_{vz} - d_{pz} f_{vx} + d_{my} f_{pw} + d_{pw} f_{my})\,\mathbf{e}_{4315}$ $+ (d_{vx} f_{py} - d_{vy} f_{px} + d_{py} f_{vx} - d_{px} f_{vy} + d_{mz} f_{pw} + d_{pw} f_{mz})\,\mathbf{e}_{4125}$ $- (d_{mx} f_{px} + d_{my} f_{py} + d_{mz} f_{pz} + d_{px} f_{mx} + d_{py} f_{my} + d_{pz} f_{mz})\,\mathbf{e}_{3215}$	

Meet Operation	Illustration
Circle where spheres **s** and **t** intersect. $\mathbf{s} \vee \mathbf{t} = (s_u t_x - s_x t_u)\,\mathbf{e}_{423} + (s_u t_y - s_y t_u)\,\mathbf{e}_{431}$ $+ (s_u t_z - s_z t_u)\,\mathbf{e}_{412} + (s_u t_w - s_w t_u)\,\mathbf{e}_{321}$ $+ (s_z t_y - s_y t_z)\,\mathbf{e}_{415} + (s_x t_z - s_z t_x)\,\mathbf{e}_{425} + (s_y t_x - s_x t_y)\,\mathbf{e}_{435}$ $+ (s_x t_w - s_w t_x)\,\mathbf{e}_{235} + (s_y t_w - s_w t_y)\,\mathbf{e}_{315} + (s_z t_w - s_w t_z)\,\mathbf{e}_{125}$	
Circle where sphere **s** and plane **g** intersect. $\mathbf{s} \vee \mathbf{g} = s_u g_x\,\mathbf{e}_{423} + s_u g_y\,\mathbf{e}_{431} + s_u g_z\,\mathbf{e}_{412} + s_u g_w\,\mathbf{e}_{321}$ $+ (s_z g_y - s_y g_z)\,\mathbf{e}_{415} + (s_x g_z - s_z g_x)\,\mathbf{e}_{425} + (s_y g_x - s_x g_y)\,\mathbf{e}_{435}$ $+ (s_x g_w - s_w g_x)\,\mathbf{e}_{235} + (s_y g_w - s_w g_y)\,\mathbf{e}_{315} + (s_z g_w - s_w g_z)\,\mathbf{e}_{125}$	
Line where planes **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\,\mathbf{e}_{415} + (g_x h_w - g_w h_x)\,\mathbf{e}_{235}$ $+ (g_x h_z - g_z h_x)\,\mathbf{e}_{425} + (g_y h_w - g_w h_y)\,\mathbf{e}_{315}$ $+ (g_y h_x - g_x h_y)\,\mathbf{e}_{435} + (g_z h_w - g_w h_z)\,\mathbf{e}_{125}$	
Dipole where sphere **s** and circle **c** intersect. $\mathbf{s} \vee \mathbf{c} = (s_y c_{gz} - s_z c_{gy} + s_u c_{vx})\,\mathbf{e}_{41} + (s_w c_{gx} - s_x c_{gw} + s_u c_{mx})\,\mathbf{e}_{23}$ $+ (s_z c_{gx} - s_x c_{gz} + s_u c_{vy})\,\mathbf{e}_{42} + (s_w c_{gy} - s_y c_{gw} + s_u c_{my})\,\mathbf{e}_{31}$ $+ (s_x c_{gy} - s_y c_{gx} + s_u c_{vz})\,\mathbf{e}_{43} + (s_w c_{gz} - s_z c_{gw} + s_u c_{mz})\,\mathbf{e}_{12}$ $+ (s_z c_{my} - s_y c_{mz} + s_w c_{vx})\,\mathbf{e}_{15} + (s_x c_{mz} - s_z c_{mx} + s_w c_{vy})\,\mathbf{e}_{25}$ $+ (s_y c_{mx} - s_x c_{my} + s_w c_{vz})\,\mathbf{e}_{35} - (s_x c_{vx} + s_y c_{vy} + s_z c_{vz})\,\mathbf{e}_{45}$	
Dipole where plane **g** and circle **c** intersect. $\mathbf{g} \vee \mathbf{c} = (g_y c_{gz} - g_z c_{gy})\,\mathbf{e}_{41} + (g_w c_{gx} - g_x c_{gw})\,\mathbf{e}_{23}$ $+ (g_z c_{gx} - g_x c_{gz})\,\mathbf{e}_{42} + (g_w c_{gy} - g_y c_{gw})\,\mathbf{e}_{31}$ $+ (g_x c_{gy} - g_y c_{gx})\,\mathbf{e}_{43} + (g_w c_{gz} - g_z c_{gw})\,\mathbf{e}_{12}$ $+ (g_z c_{my} - g_y c_{mz} + g_w c_{vx})\,\mathbf{e}_{15} + (g_x c_{mz} - g_z c_{mx} + g_w c_{vy})\,\mathbf{e}_{25}$ $+ (g_y c_{mx} - g_x c_{my} + g_w c_{vz})\,\mathbf{e}_{35} - (g_x c_{vx} + g_y c_{vy} + g_z c_{vz})\,\mathbf{e}_{45}$	
Dipole where sphere **s** and line ***l*** intersect. $\mathbf{s} \vee \boldsymbol{l} = s_u l_{vx}\,\mathbf{e}_{41} + s_u l_{vy}\,\mathbf{e}_{42} + s_u l_{vz}\,\mathbf{e}_{43}$ $+ s_u l_{mx}\,\mathbf{e}_{23} + s_u l_{my}\,\mathbf{e}_{31} + s_u l_{mz}\,\mathbf{e}_{12}$ $+ (s_z l_{my} - s_y l_{mz} + s_w l_{vx})\,\mathbf{e}_{15} + (s_x l_{mz} - s_z l_{mx} + s_w l_{vy})\,\mathbf{e}_{25}$ $+ (s_y l_{mx} - s_x l_{my} + s_w l_{vz})\,\mathbf{e}_{35} - (s_x l_{vx} + s_y l_{vy} + s_z l_{vz})\,\mathbf{e}_{45}$	
Flat point where plane **g** and line ***l*** intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\,\mathbf{e}_{15} + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\,\mathbf{e}_{25}$ $+ (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\,\mathbf{e}_{35} - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\,\mathbf{e}_{45}$	

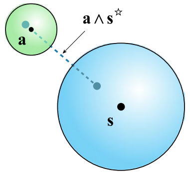
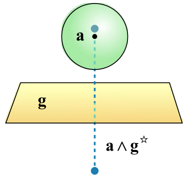
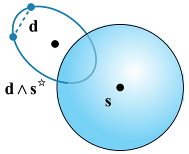
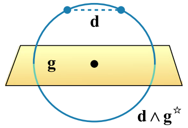
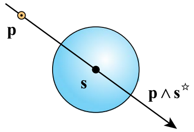
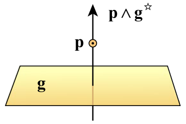
Meet Operation	Illustration
Round point contained by circles **c** and **o**. $\mathbf{c} \vee \mathbf{o} = (c_{gz} o_{my} - c_{gy} o_{mz} + c_{my} o_{gz} - c_{mz} o_{gy} + c_{vx} o_{gw} + g_{gw} o_{vx})\,\mathbf{e}_1$ $+ (c_{gx} o_{mz} - c_{gz} o_{mx} + c_{mz} o_{gx} - c_{mx} o_{gz} + c_{vy} o_{gw} + g_{gw} o_{vy})\,\mathbf{e}_2$ $+ (c_{gy} o_{mx} - c_{gx} o_{my} + c_{mx} o_{gy} - c_{my} o_{gx} + c_{vz} o_{gw} + g_{gw} o_{vz})\,\mathbf{e}_3$ $- (c_{gx} o_{vx} + c_{gy} o_{vy} + c_{gz} o_{vz} + c_{vx} o_{gx} + c_{vy} o_{gy} + c_{vz} o_{gz})\,\mathbf{e}_4$ $- (c_{mx} o_{vx} + c_{my} o_{vy} + c_{mz} o_{vz} + c_{vx} o_{mx} + c_{vy} o_{my} + c_{vz} o_{mz})\,\mathbf{e}_5$	
Round point centered on line ***l*** and contained by circle **c**. $\mathbf{c} \vee \boldsymbol{l} = (c_{gz} l_{my} - c_{gy} l_{mz} + c_{gw} l_{vx})\,\mathbf{e}_1 + (c_{gx} l_{mz} - c_{gz} l_{mx} + c_{gw} l_{vy})\,\mathbf{e}_2$ $+ (c_{gy} l_{mx} - c_{gx} l_{my} + c_{gw} l_{vz})\,\mathbf{e}_3 - (c_{gx} l_{vx} + c_{gy} l_{vy} + c_{gz} l_{vz})\,\mathbf{e}_4$ $- (c_{mx} l_{vx} + c_{my} l_{vy} + c_{mz} l_{vz} + c_{vx} l_{mx} + c_{vy} l_{my} + c_{vz} l_{mz})\,\mathbf{e}_5$	
Round point contained by sphere **s** and dipole **d**. $\mathbf{s} \vee \mathbf{d} = (s_y d_{mz} - s_z d_{my} - s_w d_{vx} + s_u d_{px})\,\mathbf{e}_1$ $+ (s_z d_{mx} - s_x d_{mz} - s_w d_{vy} + s_u d_{py})\,\mathbf{e}_2$ $+ (s_x d_{my} - s_y d_{mx} - s_w d_{vz} + s_u d_{pz})\,\mathbf{e}_3$ $+ (s_x d_{vx} + s_y d_{vy} + s_z d_{vz} + s_u d_{pw})\,\mathbf{e}_4$ $- (s_x d_{px} + s_y d_{py} + s_z d_{pz} + s_w d_{pw})\,\mathbf{e}_5$	
Round point centered in plane **g** and contained by dipole **d**. $\mathbf{g} \vee \mathbf{d} = (g_y d_{mz} - g_z d_{my} - g_w d_{vx})\,\mathbf{e}_1$ $+ (g_z d_{mx} - g_x d_{mz} - g_w d_{vy})\,\mathbf{e}_2$ $+ (g_x d_{my} - g_y d_{mx} - g_w d_{vz})\,\mathbf{e}_3$ $+ (g_x d_{vx} + g_y d_{vy} + d_z d_{vz})\,\mathbf{e}_4$ $- (g_x d_{px} + g_y d_{py} + g_z d_{pz} + g_w d_{pw})\,\mathbf{e}_5$	
Round point centered at flat point **p** and contained by sphere **s**. $\mathbf{s} \vee \mathbf{p} = s_u p_x\,\mathbf{e}_1 + s_u p_y\,\mathbf{e}_2 + s_u p_z\,\mathbf{e}_3 + s_u p_w\,\mathbf{e}_4$ $- (s_x p_x + s_y p_y + s_z p_z + s_w p_w)\,\mathbf{e}_5$	

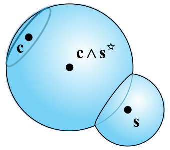
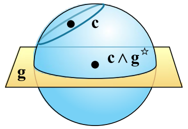
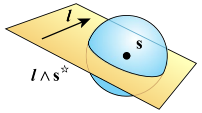
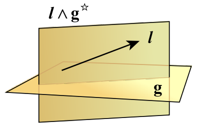
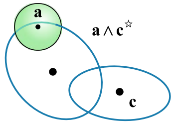
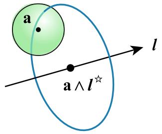
Weight Expansion

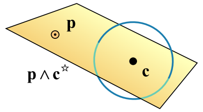
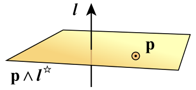
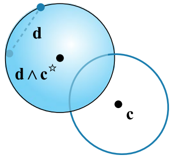
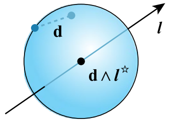
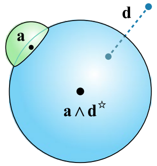
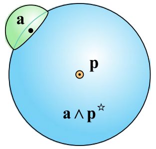
- Weight expansion calculates geometry containing one object which is orthogonal to another object

- Same math as PGA

- Projections travel along spheres, however
    - There are no ellipsoidal shapes in CGA

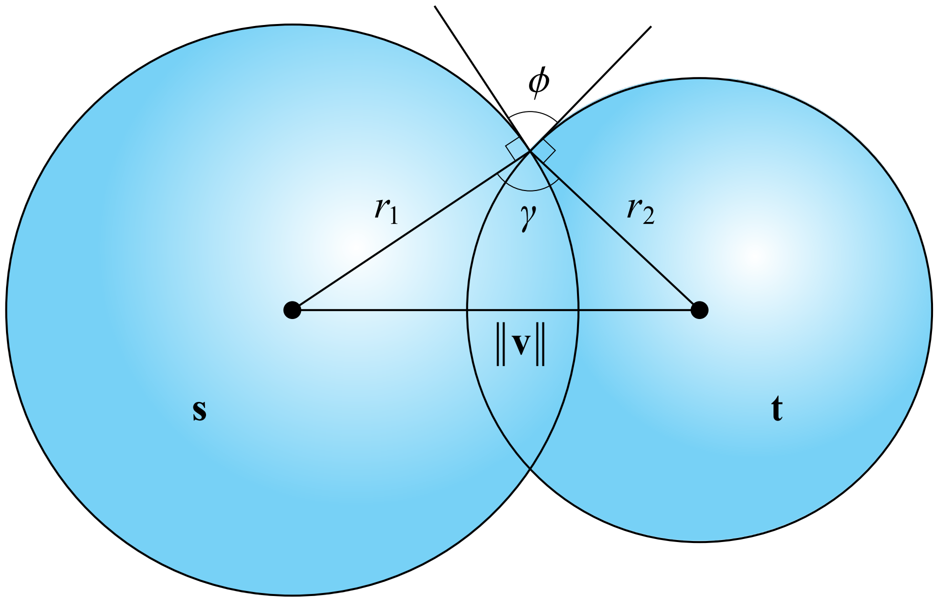
Expansion Operation	Illustration
Dipole containing round point **a** and orthogonal to sphere **s**. $\begin{aligned}\mathbf{a}\wedge\mathbf{s}^{\star}=&(a_xs_u+a_ws_x)\mathbf{e}_{41}+(a_ys_z-a_zs_y)\mathbf{e}_{23}\\&+(a_ys_u+a_ws_y)\mathbf{e}_{42}+(a_zs_x-a_xs_z)\mathbf{e}_{31}\\&+(a_zs_u+a_ws_z)\mathbf{e}_{43}+(a_xs_y-a_ys_x)\mathbf{e}_{12}\\&-(a_xs_w+a_us_x)\mathbf{e}_{15}-(a_ys_w+a_us_y)\mathbf{e}_{25}\\&-(a_zs_w+a_us_z)\mathbf{e}_{35}+(a_us_u-a_ws_w)\mathbf{e}_{45}\end{aligned}$	
Dipole containing round point **a** and orthogonal to plane **g**. $\begin{aligned}\mathbf{a}\wedge\mathbf{g}^{\star}=&a_wg_x\mathbf{e}_{41}+(a_yg_z-a_zg_y)\mathbf{e}_{23}\\&+a_wg_y\mathbf{e}_{42}+(a_zg_x-a_xg_z)\mathbf{e}_{31}\\&+a_wg_z\mathbf{e}_{43}+(a_xg_y-a_yg_x)\mathbf{e}_{12}\\&-(a_xg_w+a_ug_x)\mathbf{e}_{15}-(a_yg_w+a_ug_y)\mathbf{e}_{25}\\&-(a_zg_w+a_ug_z)\mathbf{e}_{35}-a_wg_w\mathbf{e}_{45}\end{aligned}$	
Circle containing dipole **d** and orthogonal to sphere **s**. $\begin{aligned}\mathbf{d}\wedge\mathbf{s}^{\star}=&(d_{vy}s_z-d_{vz}s_y-d_{mx}s_u)\mathbf{e}_{423}+(d_{vz}s_x-d_{vx}s_z-d_{my}s_u)\mathbf{e}_{431}\\&+(d_{vx}s_y-d_{vy}s_x-d_{mz}s_u)\mathbf{e}_{412}-(d_{mx}s_x+d_{my}s_y+d_{mz}s_z)\mathbf{e}_{321}\\&-(d_{vx}s_w+d_{pw}s_x+d_{px}s_u)\mathbf{e}_{415}+(d_{pz}s_y-d_{py}s_z-d_{mx}s_w)\mathbf{e}_{235}\\&-(d_{vy}s_w+d_{pw}s_y+d_{py}s_u)\mathbf{e}_{425}+(d_{px}s_z-d_{pz}s_x-d_{my}s_w)\mathbf{e}_{315}\\&-(d_{vz}s_w+d_{pw}s_z+d_{pz}s_u)\mathbf{e}_{435}+(d_{py}s_x-d_{px}s_y-d_{mz}s_w)\mathbf{e}_{125}\end{aligned}$	
Circle containing dipole **d** and orthogonal to plane **g**. $\begin{aligned}\mathbf{d}\wedge\mathbf{g}^{\star}=&(d_{vy}g_z-d_{vz}g_y)\mathbf{e}_{423}+(d_{vz}g_x-d_{vx}g_z)\mathbf{e}_{431}\\&+(d_{vx}g_y-d_{vy}g_x)\mathbf{e}_{412}-(d_{mx}g_x+d_{my}g_y+d_{mz}g_z)\mathbf{e}_{321}\\&-(d_{vx}g_w+d_{pw}g_x)\mathbf{e}_{415}+(d_{pz}g_y-d_{py}g_z-d_{mx}g_w)\mathbf{e}_{235}\\&-(d_{vy}g_w+d_{pw}g_y)\mathbf{e}_{425}+(d_{px}g_z-d_{pz}g_x-d_{my}g_w)\mathbf{e}_{315}\\&-(d_{vz}g_w+d_{pw}g_z)\mathbf{e}_{435}+(d_{py}g_x-d_{px}g_y-d_{mz}g_w)\mathbf{e}_{125}\end{aligned}$	
Line containing flat point **p** and orthogonal to sphere **s**. $\begin{aligned}\mathbf{p}\wedge\mathbf{s}^{\star}=&-(p_ws_x+p_xs_u)\mathbf{e}_{415}+(p_zs_y-p_ys_z)\mathbf{e}_{235}\\&-(p_ws_y+p_ys_u)\mathbf{e}_{425}+(p_xs_z-p_zs_x)\mathbf{e}_{315}\\&-(p_ws_z+p_zs_u)\mathbf{e}_{435}+(p_ys_x-p_xs_y)\mathbf{e}_{125}\end{aligned}$	
Line containing flat point **p** and orthogonal to plane **g**. $\begin{aligned}\mathbf{p}\wedge\mathbf{g}^{\star}=&-p_wg_x\mathbf{e}_{415}+(p_zg_y-p_yg_z)\mathbf{e}_{235}\\&-p_wg_y\mathbf{e}_{425}+(p_xg_z-p_zg_x)\mathbf{e}_{315}\\&-p_wg_z\mathbf{e}_{435}+(p_yg_x-p_xg_y)\mathbf{e}_{125}\end{aligned}$	

Expansion Operation	Illustration
Sphere containing circle **c** and orthogonal to sphere **s**. $\begin{aligned}\mathbf{c}\wedge\mathbf{s}^{\star}=&(c_{gw}s_u-c_{gx}s_x-c_{gy}s_y-c_{gz}s_z)\mathbf{e}_{1234}\\&+(c_{vz}s_y-c_{vy}s_z+c_{mx}s_u-c_{gx}s_w)\mathbf{e}_{4235}\\&+(c_{vx}s_z-c_{vz}s_x+c_{my}s_u-c_{gy}s_w)\mathbf{e}_{4315}\\&+(c_{vy}s_x-c_{vx}s_y+c_{mz}s_u-c_{gz}s_w)\mathbf{e}_{4125}\\&+(c_{mx}s_x+c_{my}s_y+c_{mz}s_z-c_{gw}s_w)\mathbf{e}_{3215}\end{aligned}$	
Sphere containing circle **c** and orthogonal to plane **g**. $\begin{aligned}\mathbf{c}\wedge\mathbf{g}^{\star}=&-(c_{gx}g_x+c_{gy}g_y+c_{gz}g_z)\mathbf{e}_{1234}\\&+(c_{vz}g_y-c_{vy}g_z-c_{gx}g_w)\mathbf{e}_{4235}\\&+(c_{vx}g_z-c_{vz}g_x-c_{gy}g_w)\mathbf{e}_{4315}\\&+(c_{vy}g_x-c_{vx}g_y-c_{gz}g_w)\mathbf{e}_{4125}\\&+(c_{mx}g_x+c_{my}g_y+c_{mz}g_z-c_{gw}g_w)\mathbf{e}_{3215}\end{aligned}$	
Plane containing line ***l*** and orthogonal to sphere **s**. $\begin{aligned}\boldsymbol{l}\wedge\mathbf{s}^{\star}=&(l_{vz}s_y-l_{vy}s_z+l_{mx}s_u)\mathbf{e}_{4235}+(l_{vx}s_z-l_{vz}s_x+l_{my}s_u)\mathbf{e}_{4315}\\&+(l_{vy}s_x-l_{vx}s_y+l_{mz}s_u)\mathbf{e}_{4125}+(l_{mx}s_x+l_{my}s_y+l_{mz}s_z)\mathbf{e}_{3215}\end{aligned}$	
Plane containing line ***l*** and orthogonal to plane **g**. $\begin{aligned}\boldsymbol{l}\wedge\mathbf{g}^{\star}=&(l_{vz}g_y-l_{vy}g_z)\mathbf{e}_{4235}+(l_{vx}g_z-l_{vz}g_x)\mathbf{e}_{4315}\\&+(l_{vy}g_x-l_{vx}g_y)\mathbf{e}_{4125}+(l_{mx}g_x+l_{my}g_y+l_{mz}g_z)\mathbf{e}_{3215}\end{aligned}$	
Circle containing round point **a** and orthogonal to circle **c**. $\begin{aligned}\mathbf{a}\wedge\mathbf{c}^{\star}=&(a_yc_{gz}-a_zc_{gy}-a_wc_{vx})\mathbf{e}_{423}+(a_zc_{gx}-a_xc_{gz}-a_wc_{vy})\mathbf{e}_{431}\\&+(a_xc_{gy}-a_yc_{gx}-a_wc_{vz})\mathbf{e}_{412}+(a_xc_{vx}+a_yc_{vy}+a_zc_{vz})\mathbf{e}_{321}\\&-(a_xc_{gw}+a_wc_{mx}+a_uc_{gx})\mathbf{e}_{415}+(a_zc_{my}-a_yc_{mz}-a_uc_{vx})\mathbf{e}_{235}\\&-(a_yc_{gw}+a_wc_{my}+a_uc_{gy})\mathbf{e}_{425}+(a_xc_{mz}-a_zc_{mx}-a_uc_{vy})\mathbf{e}_{315}\\&-(a_zc_{gw}+a_wc_{mz}+a_uc_{gz})\mathbf{e}_{435}+(a_yc_{mx}-a_xc_{my}-a_uc_{vz})\mathbf{e}_{125}\end{aligned}$	
Circle containing round point **a** and orthogonal to line ***l***. $\begin{aligned}\mathbf{a}\wedge\boldsymbol{l}^{\star}=&-a_wl_{vx}\mathbf{e}_{423}-a_wl_{vy}\mathbf{e}_{431}-a_wl_{vz}\mathbf{e}_{412}\\&+(a_xl_{vx}+a_yl_{vy}+a_zl_{vz})\mathbf{e}_{321}\\&-a_wl_{mx}\mathbf{e}_{415}+(a_zl_{my}-a_yl_{mz}-a_ul_{vx})\mathbf{e}_{235}\\&-a_wl_{my}\mathbf{e}_{425}+(a_xl_{mz}-a_zl_{mx}-a_ul_{vy})\mathbf{e}_{315}\\&-a_wl_{mz}\mathbf{e}_{435}+(a_yl_{mx}-a_xl_{my}-a_ul_{vz})\mathbf{e}_{125}\end{aligned}$	

Expansion Operation	Illustration
Plane containing flat point **p** and orthogonal to circle **c**. $\begin{aligned}\mathbf{p}\wedge\mathbf{c}^{\star}=&(p_yc_{gz}-p_zc_{gy}-p_wc_{vx})\mathbf{e}_{4235}\\&+(p_zc_{gx}-p_xc_{gz}-p_wc_{vy})\mathbf{e}_{4315}\\&+(p_xc_{gy}-p_yc_{gx}-p_wc_{vz})\mathbf{e}_{4125}\\&+(p_xc_{vx}+p_yc_{vy}+p_zc_{vz})\mathbf{e}_{3215}\end{aligned}$	
Plane containing flat point **p** and orthogonal to line ***l***. $\begin{aligned}\mathbf{p}\wedge\boldsymbol{l}^{\star}=&-p_wl_{vx}\mathbf{e}_{4235}-p_wl_{vy}\mathbf{e}_{4315}-p_wl_{vz}\mathbf{e}_{4125}\\&+(p_xl_{vx}+p_yl_{vy}+p_zl_{vz})\mathbf{e}_{3215}\end{aligned}$	
Sphere containing dipole **d** and orthogonal to circle **c**. $\begin{aligned}\mathbf{d}\wedge\mathbf{c}^{\star}=&(d_{vx}c_{vx}+d_{vy}c_{vy}+d_{vz}c_{vz}+d_{mx}c_{gx}+d_{my}c_{gy}+d_{mz}c_{gz})\mathbf{e}_{1234}\\&+(d_{vz}c_{my}-d_{vy}c_{mz}-d_{pw}c_{vx}+d_{py}c_{gz}-d_{pz}c_{gy}+d_{mx}c_{gw})\mathbf{e}_{4235}\\&+(d_{vx}c_{mz}-d_{vz}c_{mx}-d_{pw}c_{vy}+d_{pz}c_{gx}-d_{px}c_{gz}+d_{my}c_{gw})\mathbf{e}_{4315}\\&+(d_{vy}c_{mx}-d_{vx}c_{my}-d_{pw}c_{vz}+d_{px}c_{gy}-d_{py}c_{gx}+d_{mz}c_{gw})\mathbf{e}_{4125}\\&+(d_{px}c_{vx}+d_{py}c_{vy}+d_{pz}c_{vz}+d_{mx}c_{mx}+d_{my}c_{my}+d_{mz}c_{mz})\mathbf{e}_{3215}\end{aligned}$	
Sphere containing dipole **d** and orthogonal to line ***l***. $\begin{aligned}\mathbf{d}\wedge\boldsymbol{l}^{\star}=&(d_{vx}l_{vx}+d_{vy}l_{vy}+d_{vz}l_{vz})\mathbf{e}_{1234}\\&+(d_{vz}l_{my}-d_{vy}l_{mz}-d_{pw}l_{vx})\mathbf{e}_{4235}\\&+(d_{vx}l_{mz}-d_{vz}l_{mx}-d_{pw}l_{vy})\mathbf{e}_{4315}\\&+(d_{vy}l_{mx}-d_{vx}l_{my}-d_{pw}l_{vz})\mathbf{e}_{4125}\\&+(d_{px}l_{vx}+d_{py}l_{vy}+d_{pz}l_{vz}+d_{mx}l_{mx}+d_{my}l_{my}+d_{mz}l_{mz})\mathbf{e}_{3215}\end{aligned}$	
Sphere containing round point **a** and orthogonal to dipole **d**. $\begin{aligned}\mathbf{a}\wedge\mathbf{d}^{\star}=&(a_xd_{vx}+a_yd_{vy}+a_zd_{vz}-a_wd_{pw})\mathbf{e}_{1234}\\&+(a_zd_{my}-a_yd_{mz}+a_wd_{px}-a_ud_{vx})\mathbf{e}_{4235}\\&+(a_xd_{mz}-a_zd_{mx}+a_wd_{py}-a_ud_{vy})\mathbf{e}_{4315}\\&+(a_yd_{mx}-a_xd_{my}+a_wd_{pz}-a_ud_{vz})\mathbf{e}_{4125}\\&+(a_ud_{pw}-a_xd_{px}-a_yd_{py}-a_zd_{pz})\mathbf{e}_{3215}\end{aligned}$	
Sphere containing round point **a** and centered at flat point **p**. $\begin{aligned}\mathbf{a}\wedge\mathbf{p}^{\star}=&-a_wp_w\mathbf{e}_{1234}+a_wp_x\mathbf{e}_{4235}+a_wp_y\mathbf{e}_{4315}+a_wp_z\mathbf{e}_{4125}\\&+(a_up_w-a_xp_x-a_yp_y-a_zp_z)\mathbf{e}_{3215}\end{aligned}$	

Dot Products

- Dot product between two spheres is product of radii multiplied by cosine of angle between tangent planes where they intersect

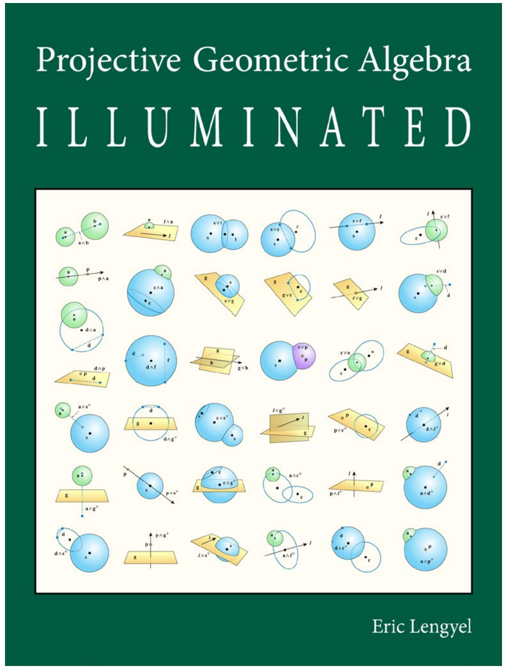


$$\mathbf{v}^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\gamma$$

$$\mathbf{s}\bullet\mathbf{t} = \tfrac{1}{2}\left(\mathbf{v}^2 - r_1^2 - r_2^2\right) = -r_1r_2\cos\gamma = r_1r_2\cos\phi$$

References

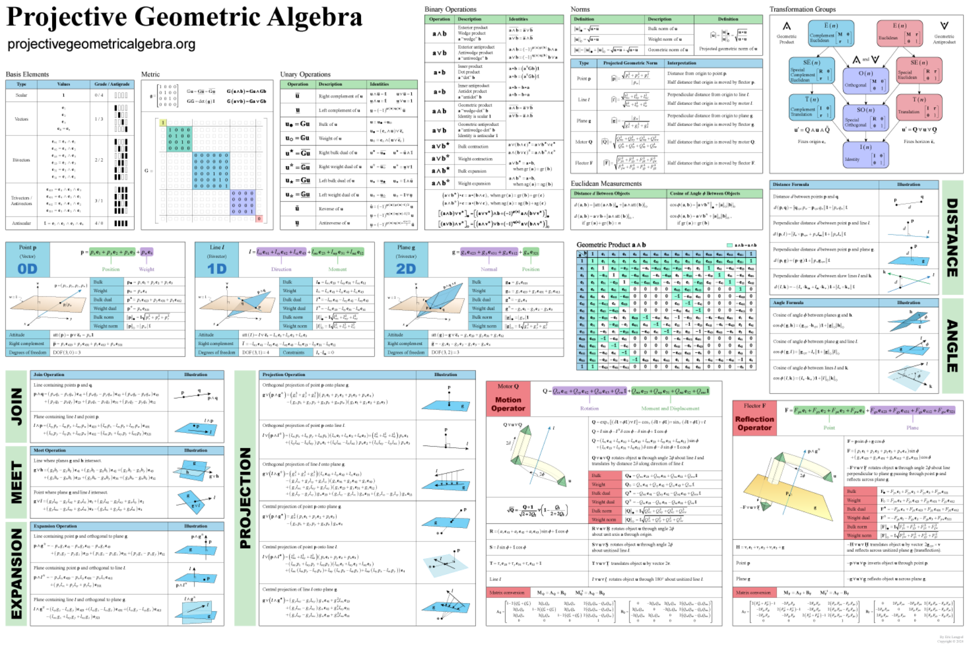
- Projective Geometric Algebra Illuminated
- projectivegeometricalgebra.org



Projective Geometric Algebra

ILLUMINATED

Eric Lengyel



Projective Geometric Algebra

projectivegeometricalgebra.org

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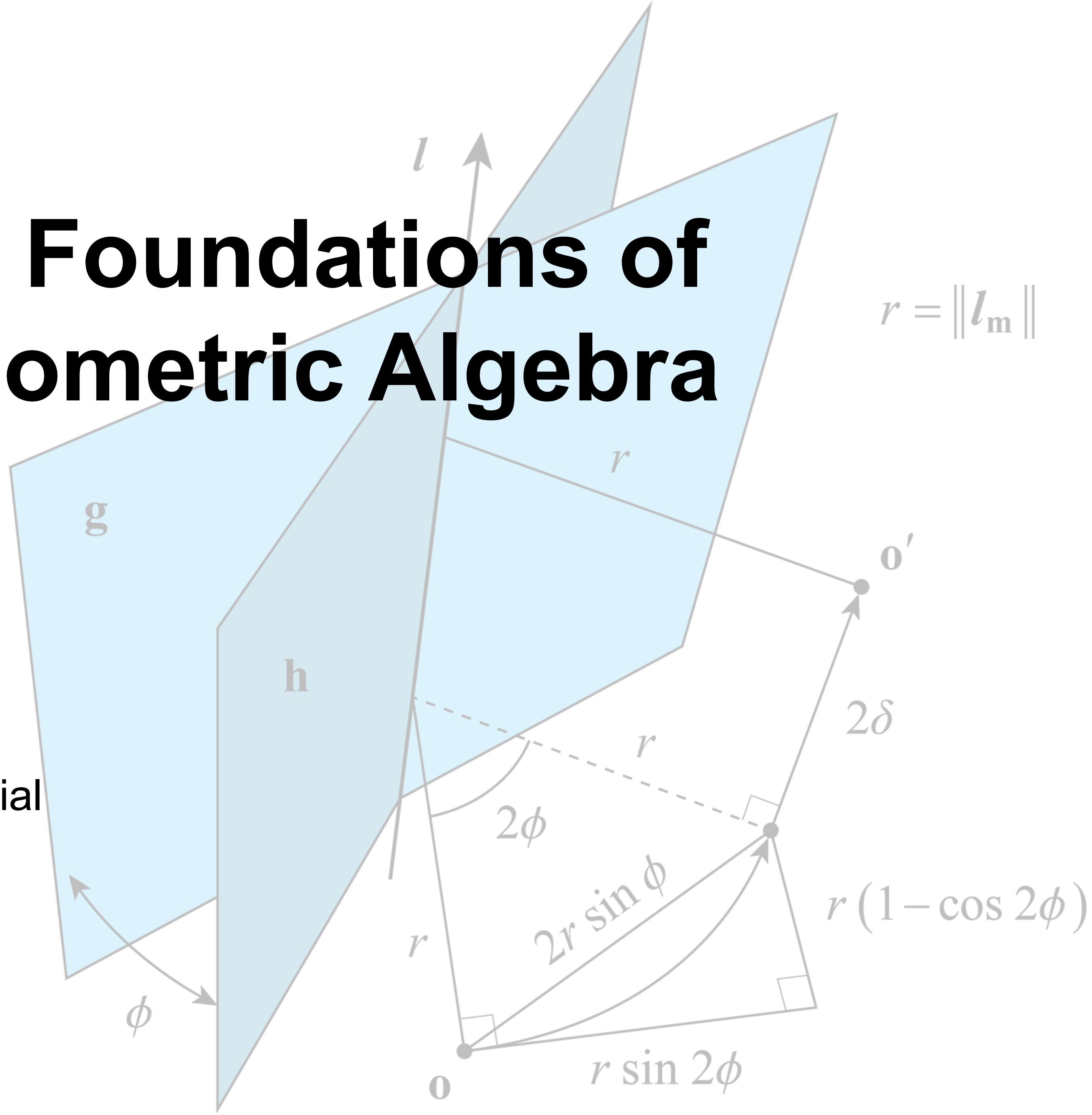
Mathematical Foundations of Projective Geometric Algebra

Eric Lengyel, Ph.D.

Space Imaging Workshop Tutorial
Georgia Tech
October 9, 2024



$\mathbf{e}_{1234}$

4D Exterior Algebra

- One scalar **1**
- Four vector basis elements
- Six bivector basis elements
- Four trivector basis elements
- One antiscalar $\mathbb{1}$

Type	**Values**	**Grade / Antigrade**	
Scalar	$\mathbf{1}$	0 / 4	▯▯▯▯
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4 = \mathbf{e}_n$	1 / 3	▮▯▯▯ ▯▮▯▯ ▯▯▮▯ ▯▯▯▮
Bivectors	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$	2 / 2	▮▯▯▮ ▯▮▯▮ ▯▯▮▮ ▯▮▮▯ ▮▯▮▯ ▮▮▯▯
Trivectors / Antivectors	$\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$	3 / 1	▯▮▮▮ ▮▯▮▮ ▮▮▯▮ ▮▮▮▯
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	▮▮▮▮

Geometric Products

- For <u>vectors</u> **a** and **b**:

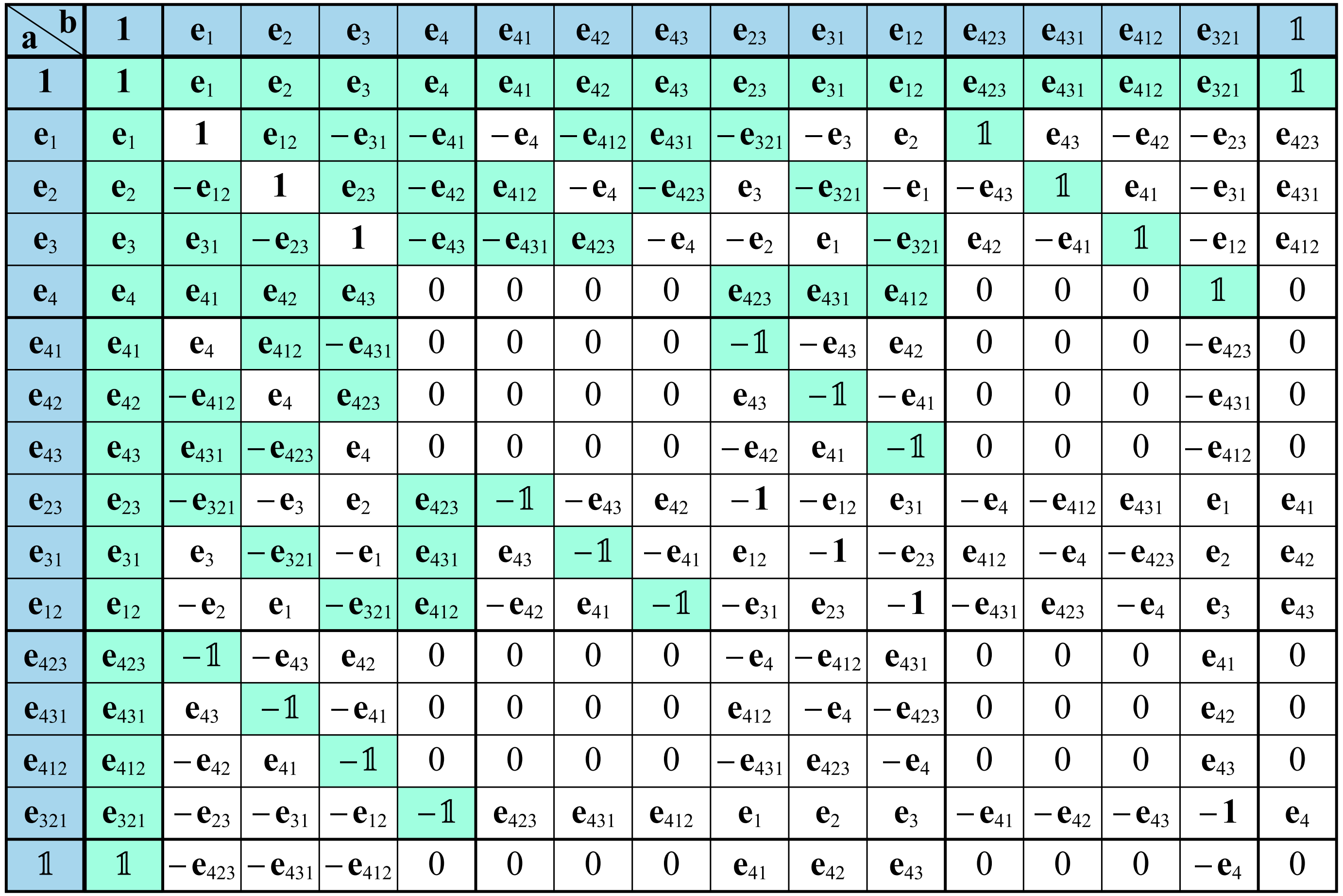
$$\mathbf{a} \mathbin{⟑} \mathbf{b} = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

- For <u>antivectors</u> **a** and **b**:

$$\mathbf{a} \mathbin{⟇} \mathbf{b} = \mathbf{a} \circ \mathbf{b} + \mathbf{a} \vee \mathbf{b}$$

4D Geometric Product

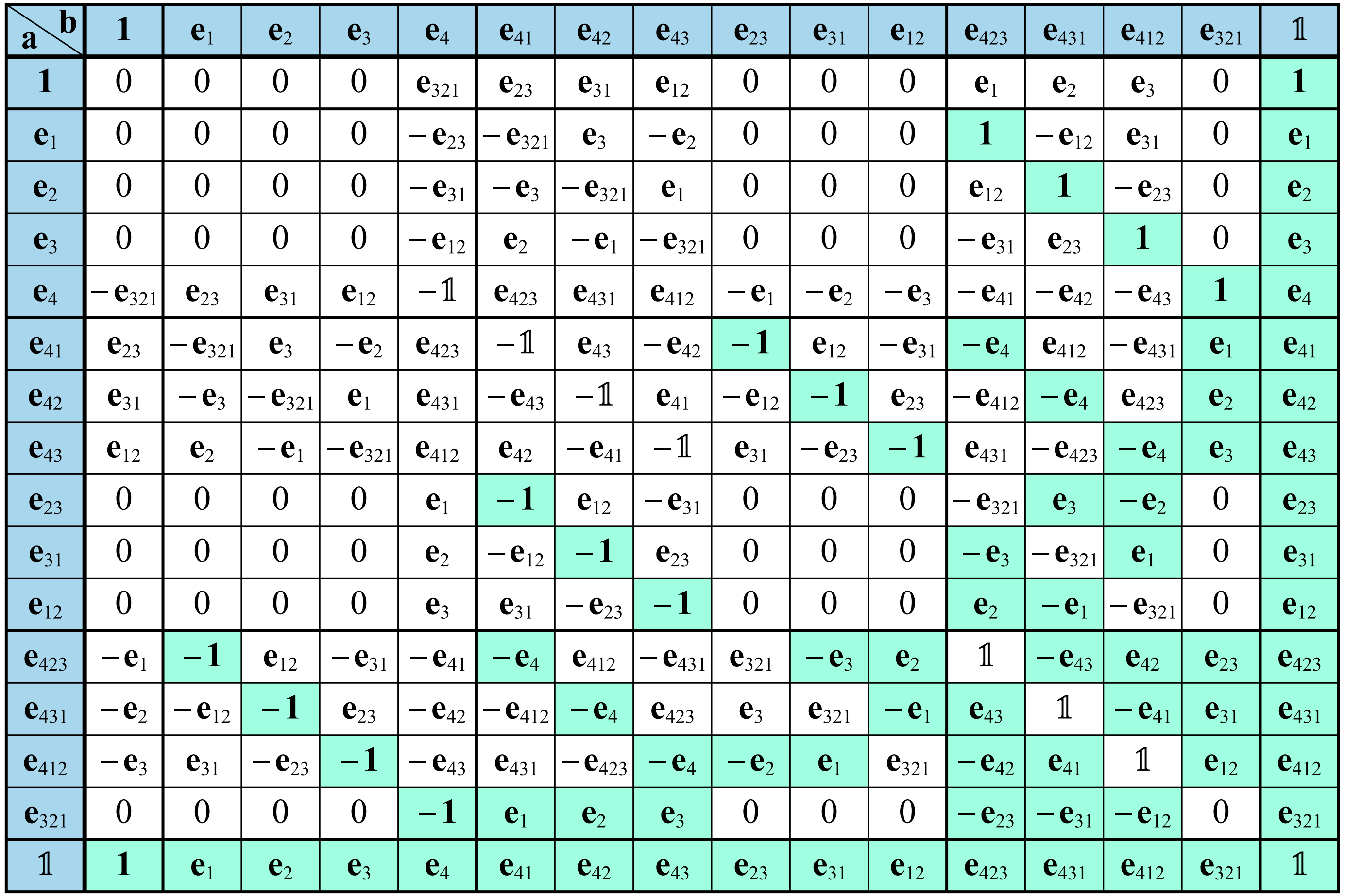
Geometric Product **a ⟑ b**



a \ b	1	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
1	1	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_{1}$	$\mathbf{e}_{1}$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{2}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{43}$	$\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{3}$	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbb{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{4}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	$-\mathbf{e}_{423}$	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{4}$	$\mathbf{e}_{423}$	0	0	0	0	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	$-\mathbf{e}_{431}$	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{4}$	0	0	0	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	$-\mathbf{e}_{412}$	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	-1	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_{1}$	$\mathbf{e}_{41}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	-1	$-\mathbf{e}_{23}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{2}$	$\mathbf{e}_{42}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	-1	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{3}$	$\mathbf{e}_{43}$
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	0	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	0	0	0	$\mathbf{e}_{41}$	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	0	0	0	$\mathbf{e}_{42}$	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	0	0	0	$\mathbf{e}_{43}$	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	-1	$\mathbf{e}_{4}$
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_{4}$	0

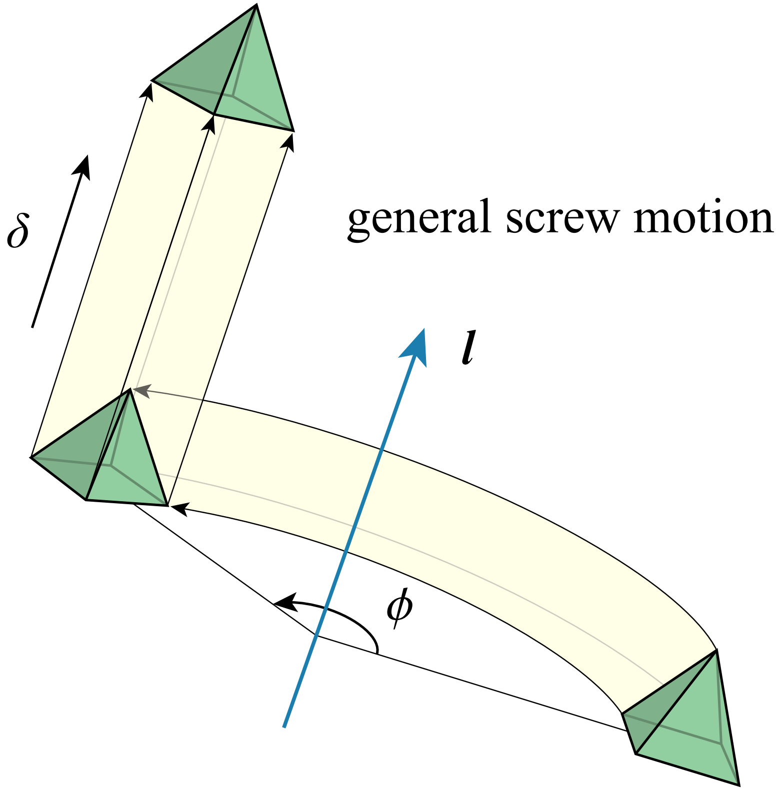
4D Geometric Antiproduct

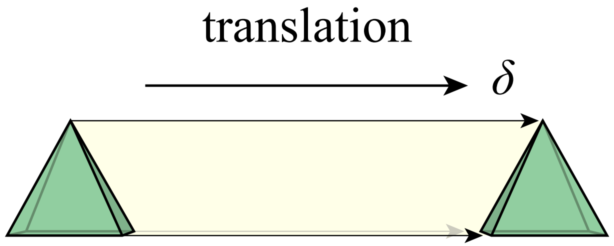
Geometric Antiproduct $\mathbf{a} ⟇ \mathbf{b}$

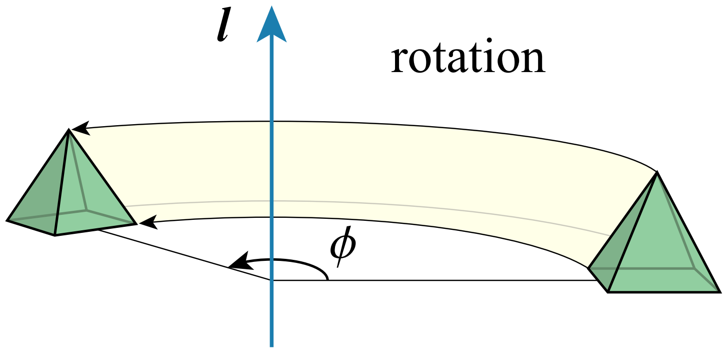


a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	0	$\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	0	0	$\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$	$-\mathbf{e}_{321}$	$\mathbf{e}_1$	0	0	0	$\mathbf{e}_{12}$	$\mathbf{1}$	$-\mathbf{e}_{23}$	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	$-\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{321}$	0	0	0	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{1}$	0	$\mathbf{e}_3$
$\mathbf{e}_4$	$-\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{1}$	$\mathbf{e}_4$
$\mathbf{e}_{41}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	$-\mathbf{e}_{321}$	$\mathbf{e}_1$	$\mathbf{e}_{431}$	$-\mathbf{e}_{43}$	$-\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_4$	$\mathbf{e}_{423}$	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbb{1}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	$\mathbf{e}_1$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	0	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	0	0	0	$-\mathbf{e}_3$	$-\mathbf{e}_{321}$	$\mathbf{e}_1$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	$-\mathbf{e}_1$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	$-\mathbf{e}_2$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_4$	$\mathbf{e}_{423}$	$\mathbf{e}_3$	$\mathbf{e}_{321}$	$-\mathbf{e}_1$	$\mathbf{e}_{43}$	$\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	$-\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{321}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbb{1}$	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$-\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

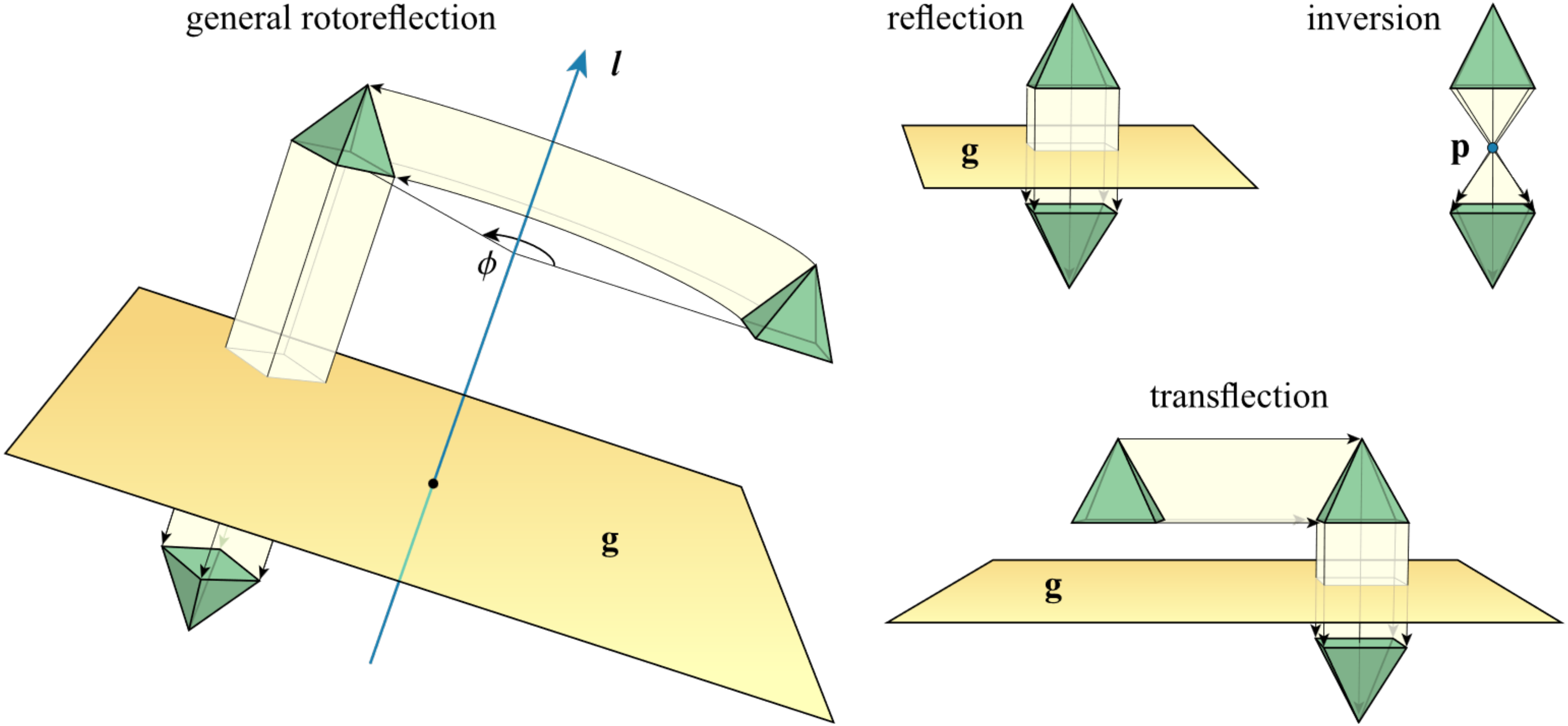
Proper Euclidean Isometries







Improper Euclidean Isometries



general rotoreflection

reflection

inversion

transflection

Plane Reflection

- Sandwich antiproduct with plane **g** performs reflection:

$$\mathbf{u}' = \mathbf{g} \mathbin{\dot\vee} \mathbf{u} \mathbin{\dot\vee} \mathbf{g}$$

- Multiple reflections stack outward from **u**:

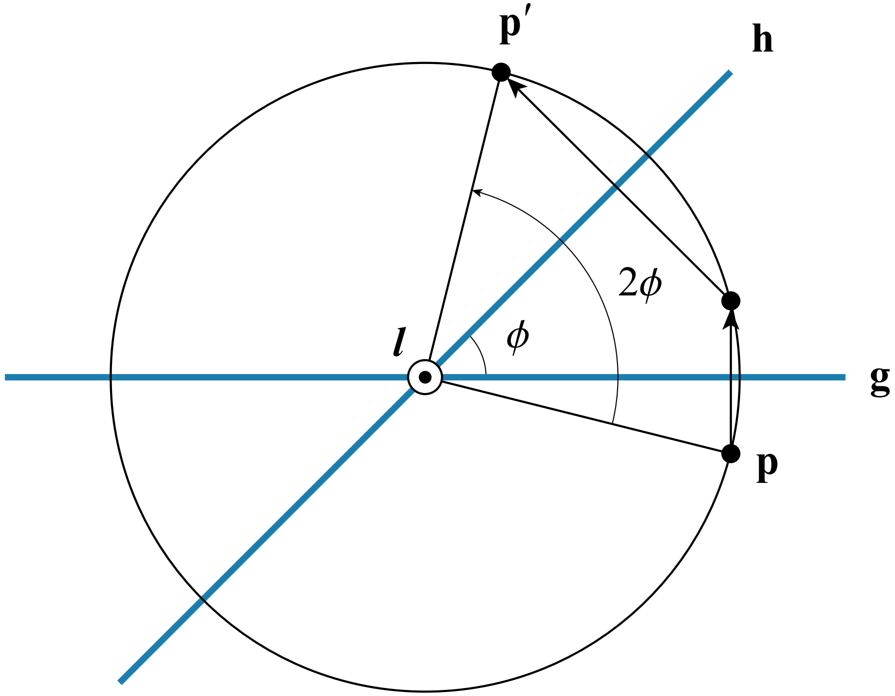
$$\mathbf{u}' = \left(\mathbf{h} \mathbin{\dot\vee} \mathbf{g}\right) \mathbin{\dot\vee} \mathbf{u} \mathbin{\dot\vee} \left(\mathbf{g} \mathbin{\dot\vee} \mathbf{h}\right)$$

- Basis for all Euclidean isometries

Rotation about a Line

- Let **g** and **h** be planes meeting at an angle ϕ
- Reflection across **g** followed by **h** is rotation through 2ϕ about line ***l*** where planes intersect

$$\boldsymbol{l} = \frac{\mathbf{h} \vee \mathbf{g}}{\left\| \mathbf{h} \vee \mathbf{g} \right\|_{\circ}}$$



Rotation about a Line

- Planes multiply together under geometric antiproduct to form rotation operator **R**

$$\mathbf{p}' = \mathbf{h} ⟇ \left(\mathbf{g} ⟇ \mathbf{p} ⟇ \mathbf{g} \right) ⟇ \mathbf{h}$$

$$\mathbf{p}' = \mathbf{R} ⟇ \mathbf{p} ⟇ \underset{\sim}{\mathbf{R}}$$

$$\mathbf{R} = \mathbf{h} ⟇ \mathbf{g}$$

Rotation about a Line

- General form of rotation operator **R**:

$$\mathbf{R} = \boldsymbol{l} \sin\phi + \mathbb{1} \cos\phi$$

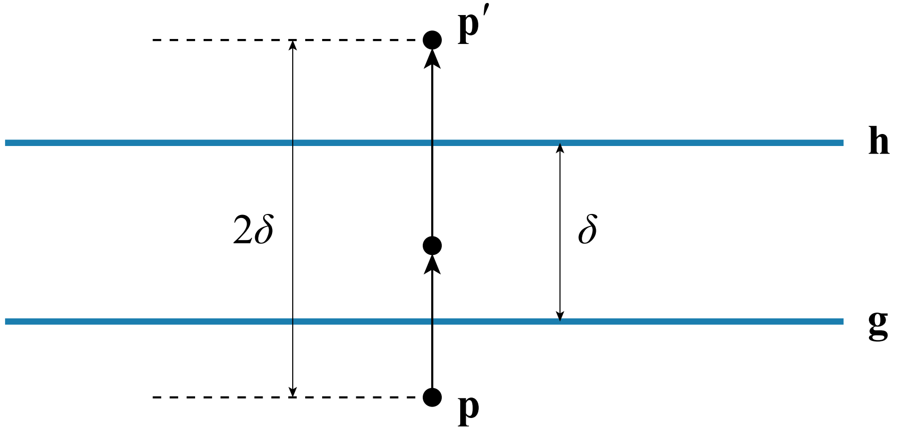
- Rotates through angle 2ϕ about unitized line ***l***

$$\mathbf{u}' = \mathbf{R} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{R}}$$

- Rotates any geometry and even other operators

Translation

- If planes **g** and **h** are parallel, result is a translation
- Translation goes along normal direction by twice the distance δ between the planes



Translation

- General form of translation operator **T**:

$$\mathbf{T} = \tau_x \mathbf{e}_{23} + \tau_y \mathbf{e}_{31} + \tau_z \mathbf{e}_{12} + \mathbb{1}$$

- Translates by displacement vector $2\boldsymbol{\tau}$

$$\mathbf{u}' = \mathbf{T} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{T}}$$

- Translates any geometry and even other operators

Motor

- General form of a motor:

$$\mathbf{Q} = \underbrace{Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1}}_{\text{Rotation Quaternion}} + \underbrace{Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}}_{\text{Moment and Displacement}}$$

- Performs any combination of rotations and translations

$$\mathbf{u}' = \mathbf{Q} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{Q}}$$

- Always true that $\mathbf{Q_v} \cdot \mathbf{Q_m} = 0$

Motor

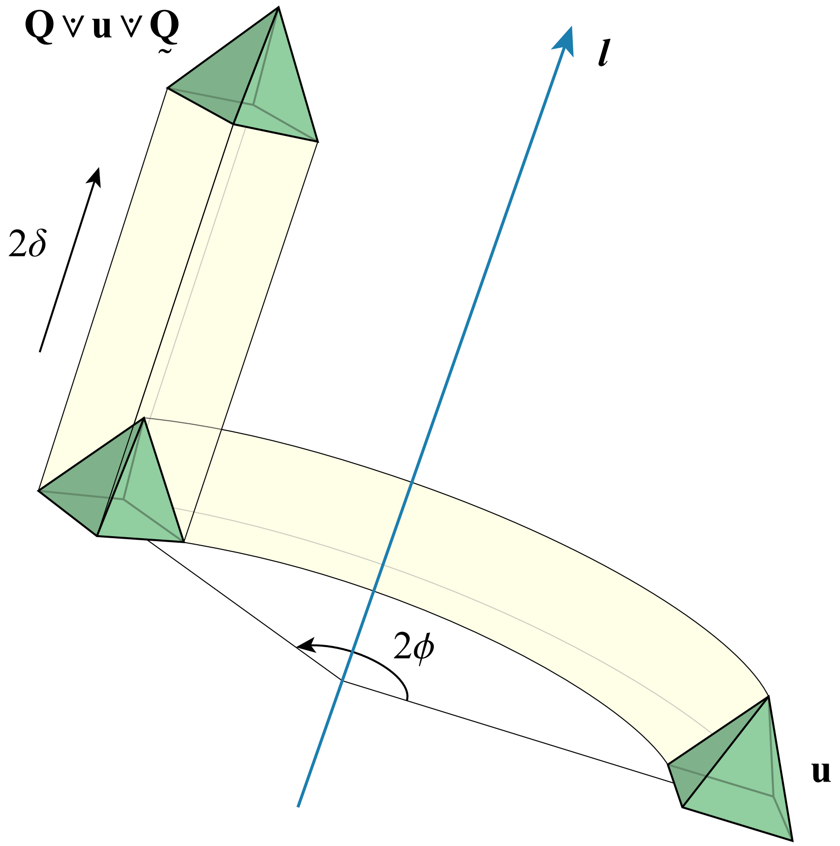
- General form of a motor:

$$\mathbf{Q} = \underbrace{Q_{vx}\mathbf{e}_{41} + Q_{vy}\mathbf{e}_{42} + Q_{vz}\mathbf{e}_{43} + Q_{vw}\mathbb{1}}_{\text{Rotation Quaternion}} + \underbrace{Q_{mx}\mathbf{e}_{23} + Q_{my}\mathbf{e}_{31} + Q_{mz}\mathbf{e}_{12} + Q_{mw}\mathbf{1}}_{\text{Moment and Displacement}}$$

- Simple motor has $Q_{mw} = 0$ and is pure rotation or translation

- Quaternion has no bulk part (green)

Motor



$$\mathbf{Q} = \exp_{\dot{\vee}}\left[\left(\delta\mathbf{1} + \varphi\mathbb{1}\right)\dot{\vee}\,\boldsymbol{l}\right] = \boldsymbol{l}\sin\varphi - \boldsymbol{l}^{\star}\delta\cos\varphi - \delta\sin\varphi + \mathbb{1}\cos\varphi$$

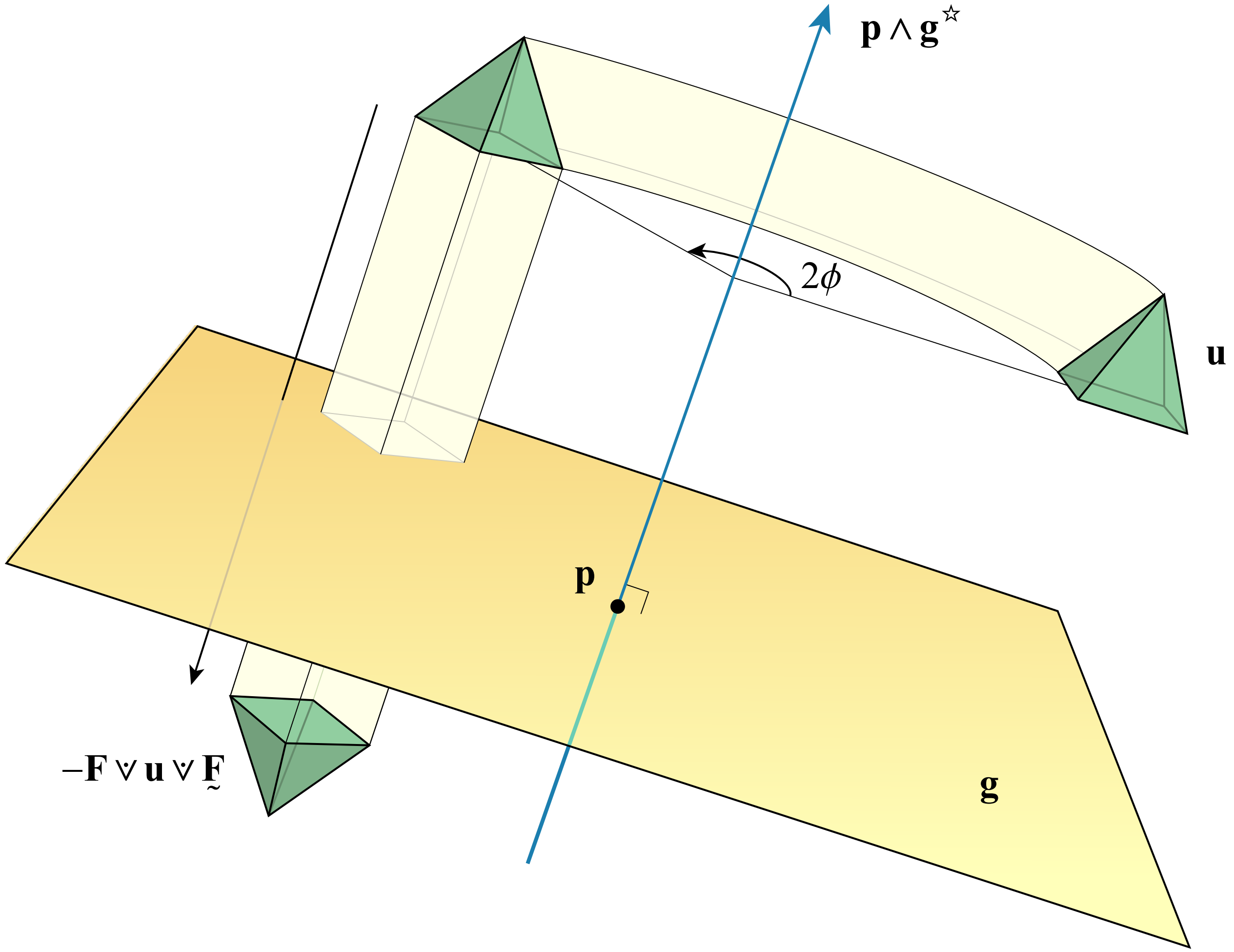
Flector

- General form of a flector:

$$\mathbf{F} = \underbrace{F_{px}\,\mathbf{e}_1 + F_{py}\,\mathbf{e}_2 + F_{pz}\,\mathbf{e}_3 + F_{pw}\,\mathbf{e}_4}_{\text{Point}} + \underbrace{F_{gx}\,\mathbf{e}_{423} + F_{gy}\,\mathbf{e}_{431} + F_{gz}\,\mathbf{e}_{412} + F_{gw}\,\mathbf{e}_{321}}_{\text{Plane}}$$

- Performs any combination of rotoreflections

Flector



$$\mathbf{F} = \mathbf{p} \sin \phi + \mathbf{g} \cos \phi$$

Motor Parameterization

- A motion operator is parameterized by:
    - A unitized line ***l***
    - A half rotation angle ϕ
    - A half displacement distance δ

- Exponential with respect to geometric antiproduct:

$$\mathbf{Q} = \exp_{\veedot}\left[\left(\delta\mathbf{1} + \phi\mathbb{1}\right) \veedot \boldsymbol{l}\right] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

- $\delta\mathbf{1} + \phi\mathbb{1}$ is *pitch* of screw transformation

Motor Parameterization

- Given arbitrary motor **Q**, can calculate parameters

$$\mathbf{Q} = Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1} + Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}$$

$$\mathbf{Q} = \exp_{\dot{\vee}}\left[(\delta\mathbf{1} + \phi\mathbb{1}) \dot{\vee} \boldsymbol{l}\right] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

$$s = \sin\phi = \sqrt{1 - Q_{vw}^2} \qquad \delta = -\frac{Q_{mw}}{s} \qquad \phi = \tan^{-1}\left(\frac{s}{Q_{vw}}\right)$$

$$\boldsymbol{l}_{\mathbf{v}} = \frac{1}{s}\mathbf{Q}_{vxyz} \qquad \boldsymbol{l}_{\mathbf{m}} = \frac{1}{s}\left(\mathbf{Q}_{mxyz} + \frac{Q_{vw}Q_{mw}}{s^2}\mathbf{Q}_{vxyz}\right)$$

Motor Interpolation

- To interpolate from motor **Q**$_1$ to motor **Q**$_2$, first calculate

$$\mathbf{Q}_0 = \mathbf{Q}_2 \mathbin{\dot\vee} \mathbf{Q}_1^{-1} = \mathbf{Q}_2 \mathbin{\dot\vee} \underset{\sim}{\mathbf{Q}}_1$$

- Then calculate parameters ***l***, δ, and ϕ for **Q**$_0$

- Interpolate from identity $\mathbb{1}$ to **Q**$_0$ with

$$\mathbf{Q}(t) = \exp_{\dot\vee}\left[t\left(\delta\mathbf{1} + \phi\mathbb{1}\right) \mathbin{\dot\vee} \boldsymbol{l}\right] = \boldsymbol{l}\sin(t\phi) - \boldsymbol{l}^{\star} t\delta\cos(t\phi) - t\delta\sin(t\phi) + \mathbb{1}\cos(t\phi)$$

- Finally, calculate $\mathbf{Q}(t) \mathbin{\dot\vee} \mathbf{Q}_1$

Motor Interpolation

- That can be computationally expensive

- Approximate interpolation is often acceptable:

$$\mathbf{Q}(t)=(1-t)\,\mathbf{Q}_1+t\mathbf{Q}_2$$

- This needs to be unitized and constrained

$$\frac{\mathbf{Q}}{\left\|\mathbf{Q}_\mathbf{v}\right\|}\,\dot{\vee}\left(-\frac{\mathbf{Q}_\mathbf{v}\cdot\mathbf{Q}_\mathbf{m}}{\mathbf{Q}_\mathbf{v}^2}\mathbf{1}+\mathbb{1}\right)=\frac{1}{\left\|\mathbf{Q}_\mathbf{v}\right\|}\left[\mathbf{Q}-\frac{\mathbf{Q}_\mathbf{v}\cdot\mathbf{Q}_\mathbf{m}}{\mathbf{Q}_\mathbf{v}^2}\left(Q_{vx}\,\mathbf{e}_{23}+Q_{vy}\,\mathbf{e}_{31}+Q_{vz}\,\mathbf{e}_{12}+Q_{vw}\right)\right]$$

Square Root of Motor

- Special case of interpolation from 𝟙 to **Q** when *t* = 1/2

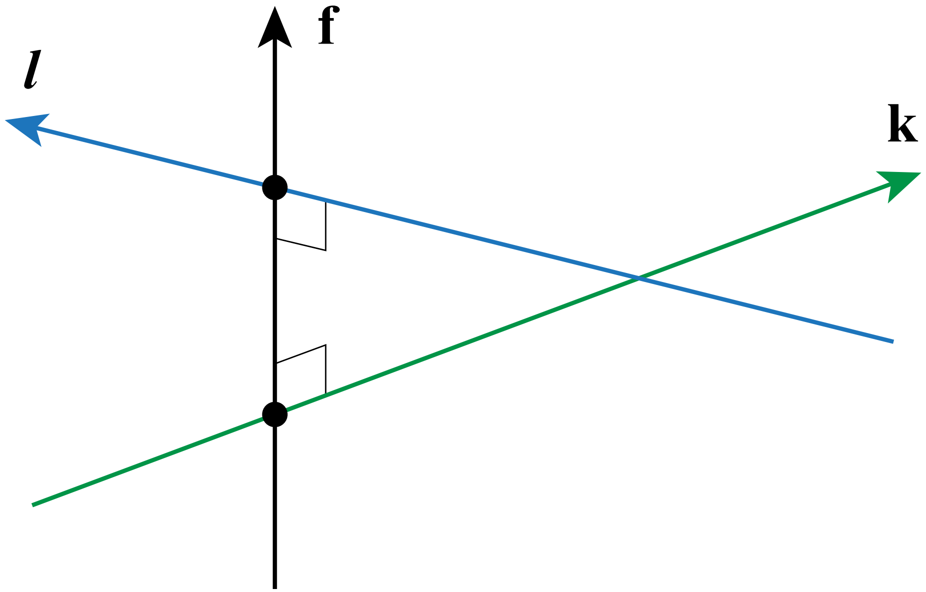
$$\sqrt[\dot{\vee}]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\sqrt{2 + 2Q_{\mathbb{1}}}} \mathbin{\dot{\vee}} \left(\mathbb{1} - \frac{Q_{\mathbf{1}}}{2 + 2Q_{\mathbb{1}}} \mathbf{1} \right)$$

- For simple motor (pure rotation or translation), this simplifies:

$$\sqrt[\dot{\vee}]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\left\| \mathbf{Q} + \mathbb{1} \right\|_{\bigcirc}}$$

Line to Line Motion

- Let **k** and ***l*** be lines separated by distance δ with angle ϕ between directions
- Operator $\boldsymbol{l} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{k}}$ rotates by 2ϕ and translates by distance 2δ about line **f** connecting closest points
- Square root of this operator transforms line **k** into line ***l***



Sec. 3.6.3

Motor-Point Transformation

- 25 multiply-adds:

$$\mathbf{p}'_{xyz} = \mathbf{p}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v}\times\mathbf{a} - Q_{mw}\,p_w\,\mathbf{v}\right)$$

$$p'_w = p_w$$

$$\mathbf{a} = \mathbf{v}\times\mathbf{p}_{xyz} + p_w\mathbf{m}$$

$$\mathbf{v} = \left(Q_{vx}, Q_{vy}, Q_{vz}\right)$$

$$\mathbf{m} = \left(Q_{mx}, Q_{my}, Q_{mz}\right)$$

- 3×4 matrix transformation only requires 12 multiply-adds, (or just 9 if $p_w = 1$)

Motor-Line Transformation

- 54 multiply-adds:

$$\boldsymbol{l}'_{\mathbf{v}} = \boldsymbol{l}_{\mathbf{v}} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a}\right)$$

$$\boldsymbol{l}'_{\mathbf{m}} = \boldsymbol{l}_{\mathbf{m}} + 2\left[Q_{mw}\mathbf{a} + Q_{vw}\left(\mathbf{b} + \mathbf{c}\right) + \mathbf{v} \times \left(\mathbf{b} + \mathbf{c}\right) + \mathbf{m} \times \mathbf{a}\right]$$

$$\mathbf{a} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{v}} \qquad \mathbf{b} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{m}} \qquad \mathbf{c} = \mathbf{m} \times \boldsymbol{l}_{\mathbf{v}}$$

- 6×6 matrix transformation only requires 27 multiply-adds

Motor-Plane Transformation

- 35 multiply-adds:

$$\mathbf{g}'_{xyz} = \mathbf{g}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a}\right)$$

$$g'_{w} = g_{w} + 2\left[\left(\mathbf{m} \times \mathbf{g}_{xyz} + Q_{mw}\mathbf{g}_{xyz}\right) \cdot \mathbf{v} - Q_{vw}\left(\mathbf{m} \cdot \mathbf{g}_{xyz}\right)\right]$$

$$\mathbf{a} = \mathbf{v} \times \mathbf{g}_{xyz}$$

- 4×4 matrix transformation only requires 13 multiply-adds

Motor to Matrix

$$\mathbf{A_Q} = \begin{bmatrix} 1-2\left(Q_{vy}^2 + Q_{vz}^2\right) & 2Q_{vx}Q_{vy} & 2Q_{vz}Q_{vx} & 2\left(Q_{vy}Q_{mz} - Q_{vz}Q_{my}\right) \\ 2Q_{vx}Q_{vy} & 1-2\left(Q_{vz}^2 + Q_{vx}^2\right) & 2Q_{vy}Q_{vz} & 2\left(Q_{vz}Q_{mx} - Q_{vx}Q_{mz}\right) \\ 2Q_{vz}Q_{vx} & 2Q_{vy}Q_{vz} & 1-2\left(Q_{vx}^2 + Q_{vy}^2\right) & 2\left(Q_{vx}Q_{my} - Q_{vy}Q_{mx}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B_Q} = \begin{bmatrix} 0 & -2Q_{vz}Q_{vw} & 2Q_{vy}Q_{vw} & 2\left(Q_{vw}Q_{mx} - Q_{vx}Q_{mw}\right) \\ 2Q_{vz}Q_{vw} & 0 & -2Q_{vx}Q_{vw} & 2\left(Q_{vw}Q_{my} - Q_{vy}Q_{mw}\right) \\ -2Q_{vy}Q_{vw} & 2Q_{vx}Q_{vw} & 0 & 2\left(Q_{vw}Q_{mz} - Q_{vz}Q_{mw}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M_Q} = \mathbf{A_Q} + \mathbf{B_Q} \qquad \mathbf{M_Q^{-1}} = \mathbf{A_Q} - \mathbf{B_Q}$$

Motor Composition

- 48 multiply-adds:

$$
\begin{aligned}
\mathbf{Q} ⟇ \mathbf{R} = {} & (Q_{vw}R_{vx} + Q_{vx}R_{vw} + Q_{vy}R_{vz} - Q_{vz}R_{vy})\,\mathbf{e}_{41} \\
& + (Q_{vw}R_{vy} - Q_{vx}R_{vz} + Q_{vy}R_{vw} + Q_{vz}R_{vx})\,\mathbf{e}_{42} \\
& + (Q_{vw}R_{vz} + Q_{vx}R_{vy} - Q_{vy}R_{vx} + Q_{vz}R_{vw})\,\mathbf{e}_{43} \\
& + (Q_{vw}R_{vw} - Q_{vx}R_{vx} - Q_{vy}R_{vy} - Q_{vz}R_{vz})\,\mathbb{1} \\
& + (Q_{mw}R_{vx} + Q_{mx}R_{vw} + Q_{my}R_{vz} - Q_{mz}R_{vy} + Q_{vw}R_{mx} + Q_{vx}R_{mw} + Q_{vy}R_{mz} - Q_{vz}R_{my})\,\mathbf{e}_{23} \\
& + (Q_{mw}R_{vy} - Q_{mx}R_{vz} + Q_{my}R_{vw} + Q_{mz}R_{vx} + Q_{vw}R_{my} - Q_{vx}R_{mz} + Q_{vy}R_{mw} + Q_{vz}R_{mx})\,\mathbf{e}_{31} \\
& + (Q_{mw}R_{vz} + Q_{mx}R_{vy} - Q_{my}R_{vx} + Q_{mz}R_{vw} + Q_{vw}R_{mz} + Q_{vx}R_{my} - Q_{vy}R_{mx} + Q_{vz}R_{mw})\,\mathbf{e}_{12} \\
& + (Q_{mw}R_{vw} - Q_{mx}R_{vx} - Q_{my}R_{vy} - Q_{mz}R_{vz} + Q_{vw}R_{mw} - Q_{vx}R_{mx} - Q_{vy}R_{my} - Q_{vz}R_{mz})\,\mathbf{1}
\end{aligned}
$$

- Composition of equivalent 3×4 matrices requires 33 multiply-adds

Matrix Advantages

- Can represent more transformations
- Can read off origin and axis directions in transformed space
- Faster to transform objects
- Faster to compose

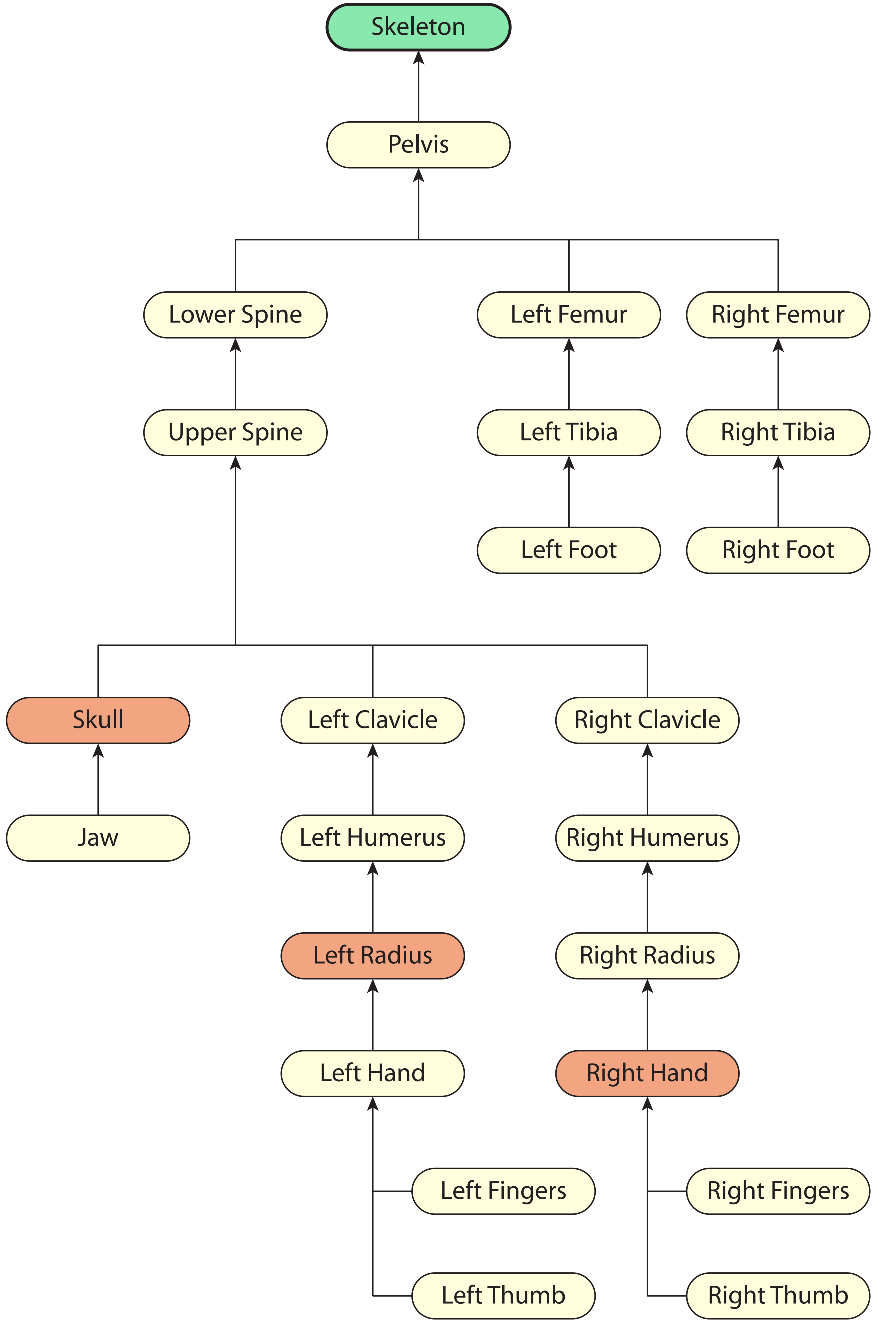
Motor Advantages

- Smaller storage requirements
  - Usually 8 floats, but can reduce to 6

- Inversion is trivial
  - Just reverse, negating six bivector components

- Better parameterization

- Better interpolation properties



- Skeleton
  - Pelvis
    - Lower Spine
      - Upper Spine
        - Skull
          - Jaw
        - Left Clavicle
          - Left Humerus
            - Left Radius
              - Left Hand
                - Left Fingers
                - Left Thumb
        - Right Clavicle
          - Right Humerus
            - Right Radius
              - Right Hand
                - Right Fingers
                - Right Thumb
    - Left Femur
      - Left Tibia
        - Left Foot
    - Right Femur
      - Right Tibia
        - Right Foot

Motor and Matrix

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_x \\ m_{21} & m_{22} & m_{23} & t_y \\ m_{31} & m_{32} & m_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Q} = Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1} + Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}$$

$$\mathbf{M} = \begin{bmatrix} 1-2\left(Q_{vy}^2 + Q_{vz}^2\right) & 2\left(Q_{vx}Q_{vy} - Q_{vz}Q_{vw}\right) & 2\left(Q_{vz}Q_{vx} + Q_{vy}Q_{vw}\right) & 2\left(Q_{vy}Q_{mz} - Q_{vz}Q_{my} + Q_{vw}Q_{mx} - Q_{vx}Q_{mw}\right) \\ 2\left(Q_{vx}Q_{vy} + Q_{vz}Q_{vw}\right) & 1-2\left(Q_{vy}^2 + Q_{vz}^2\right) & 2\left(Q_{vy}Q_{vz} - Q_{vx}Q_{vw}\right) & 2\left(Q_{vz}Q_{mx} - Q_{vx}Q_{mz} + Q_{vw}Q_{my} - Q_{vy}Q_{mw}\right) \\ 2\left(Q_{vz}Q_{vx} - Q_{vy}Q_{vw}\right) & 2\left(Q_{vy}Q_{vz} + Q_{vx}Q_{vw}\right) & 1-2\left(Q_{vx}^2 + Q_{vy}^2\right) & 2\left(Q_{vx}Q_{my} - Q_{vy}Q_{mx} + Q_{vw}Q_{mz} - Q_{vz}Q_{mw}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Dual Quaternion Skinning

Operator Duality

- All of the Euclidean isometries performed with antiproduct:

$$\mathbf{u}' = \mathbf{Q} \veedot \mathbf{u} \veedot \underset{\sim}{\mathbf{Q}}$$

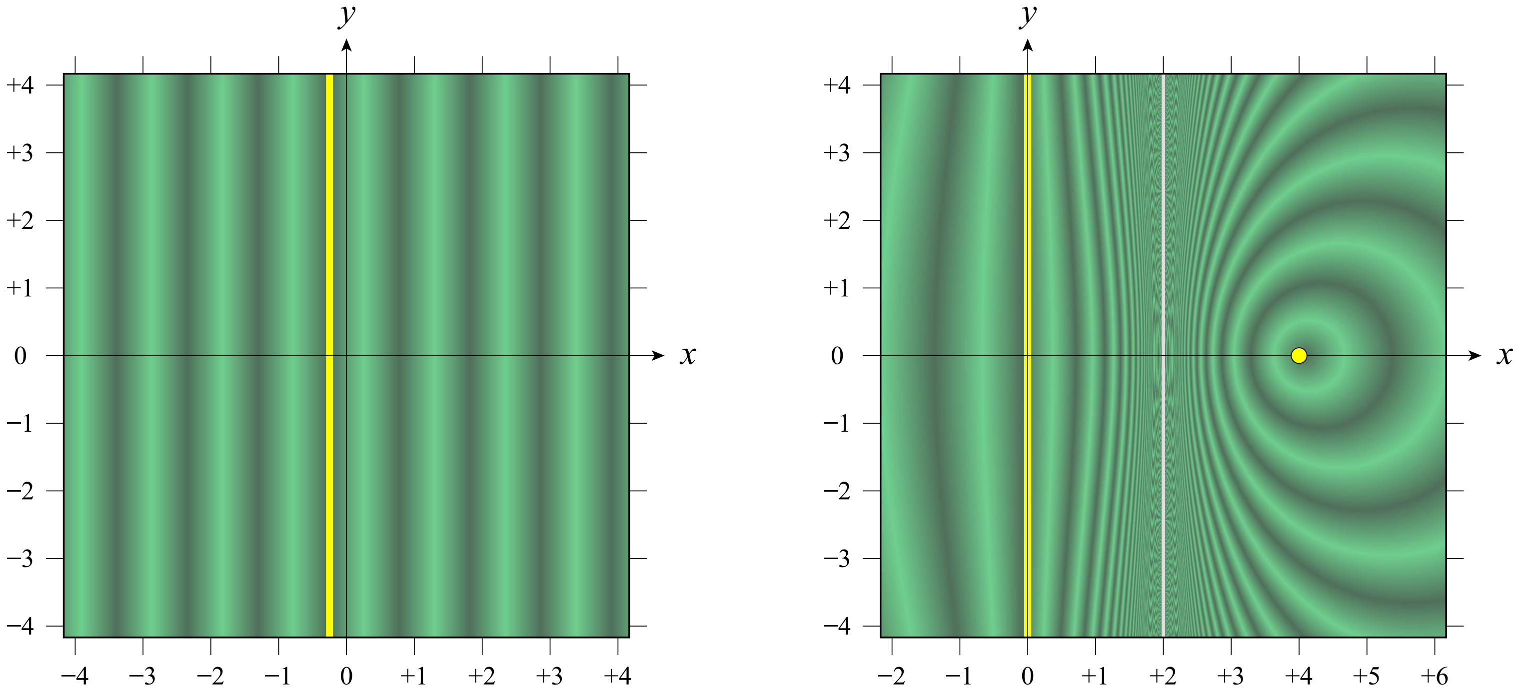
- This fixes the horizon, as required

- Separate motions occur in antispace that fix the origin

- Swapping product and antiproduct also swaps motions

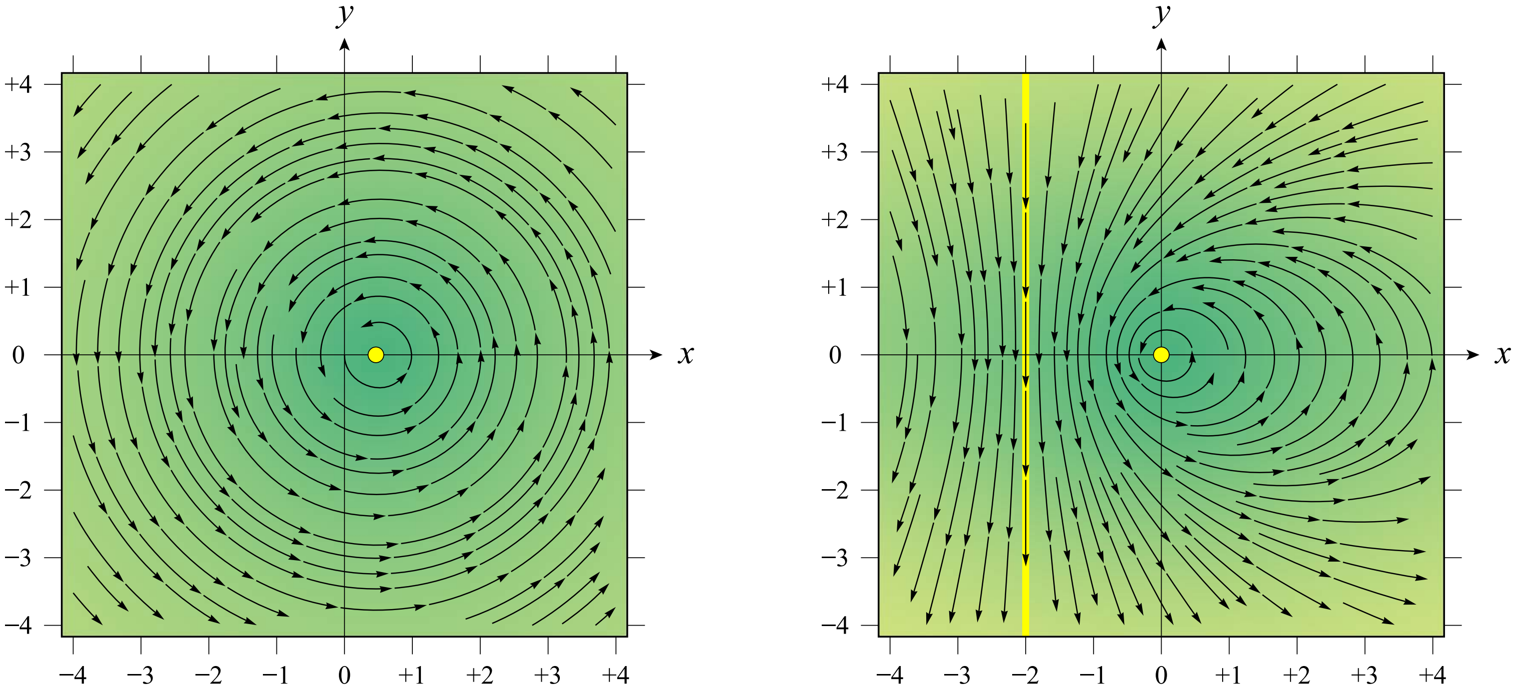
Complement Isometries

- Sandwiches with the geometric product perform *complement isometries*
- A plane reflection and its complement look like this:



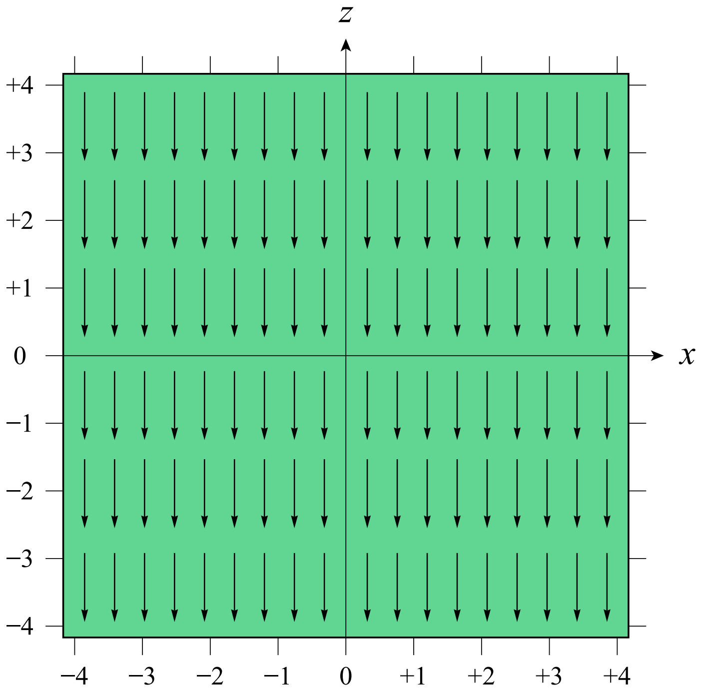
Complement Rotation

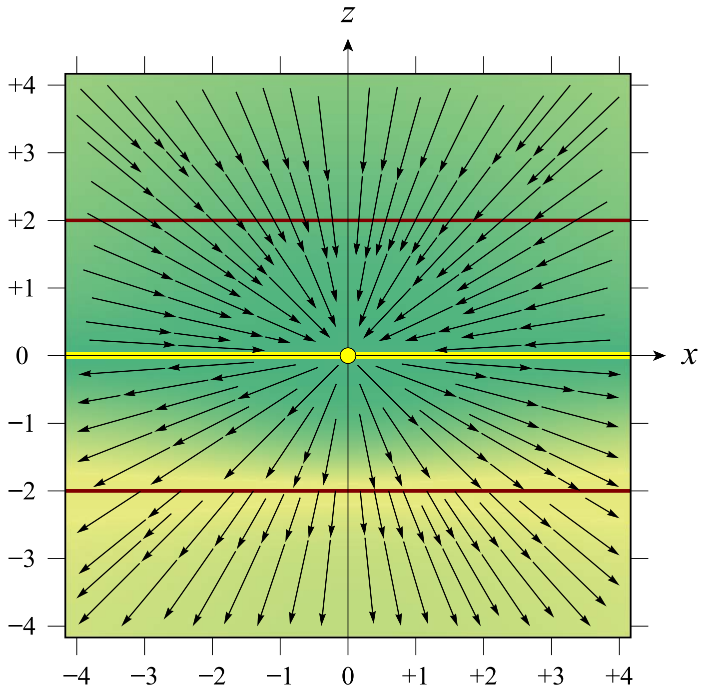
- A complement rotation moves points along paths of constant eccentricity with respect to a focus at the origin and a directrix given by a line



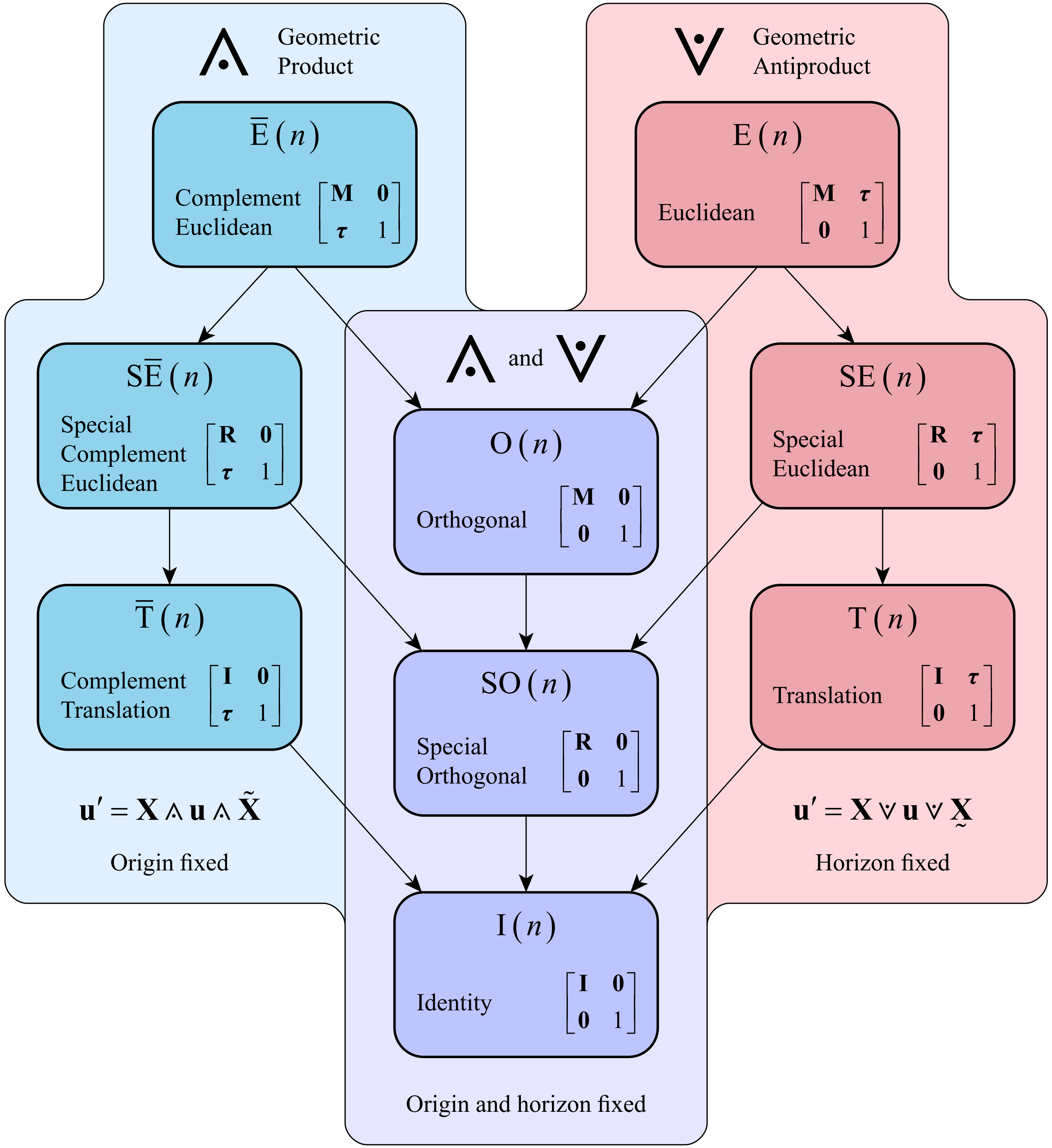
Complement Translation

- A complement translation performs a *perspective projection* having a specific direction and focal length





Transformation Groups



⟑ Geometric Product

$\overline{\mathrm{E}}(n)$

Complement Euclidean $\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \boldsymbol{\tau} & 1 \end{bmatrix}$

$\mathrm{S}\overline{\mathrm{E}}(n)$

Special Complement Euclidean $\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \boldsymbol{\tau} & 1 \end{bmatrix}$

$\overline{\mathrm{T}}(n)$

Complement Translation $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \boldsymbol{\tau} & 1 \end{bmatrix}$

$\mathbf{u}' = \mathbf{X} ⟑ \mathbf{u} ⟑ \tilde{\mathbf{X}}$

Origin fixed

⟑ and ⟇

$\mathrm{O}(n)$

Orthogonal $\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathrm{SO}(n)$

Special Orthogonal $\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathrm{I}(n)$

Identity $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$

Origin and horizon fixed

⟇ Geometric Antiproduct

$\mathrm{E}(n)$

Euclidean $\begin{bmatrix} \mathbf{M} & \boldsymbol{\tau} \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathrm{SE}(n)$

Special Euclidean $\begin{bmatrix} \mathbf{R} & \boldsymbol{\tau} \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathrm{T}(n)$

Translation $\begin{bmatrix} \mathbf{I} & \boldsymbol{\tau} \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathbf{u}' = \mathbf{X} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{X}}$

Horizon fixed

Common Subgroups

- Operators fixing both the origin and the horizon have two forms

- In particular, quaternions exist in both $\mathrm{SE}(3)$ and $\mathrm{S\overline{E}}(3)$

$$\mathbf{q} = q_x \mathbf{e}_{41} + q_y \mathbf{e}_{42} + q_z \mathbf{e}_{43} + q_w \mathbb{1} \qquad \mathbf{u}' = \mathbf{q} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \utilde{\mathbf{q}}$$

$$\mathbf{q} = -q_x \mathbf{e}_{23} - q_y \mathbf{e}_{31} - q_z \mathbf{e}_{12} + q_w \mathbf{1} \qquad \mathbf{u}' = \mathbf{q} \mathbin{\dot{\wedge}} \mathbf{u} \mathbin{\dot{\wedge}} \tilde{\mathbf{q}}$$

Conformal Motions

- All motions of PGA transfer to CGA with factor of $\mathbf{e}_5$

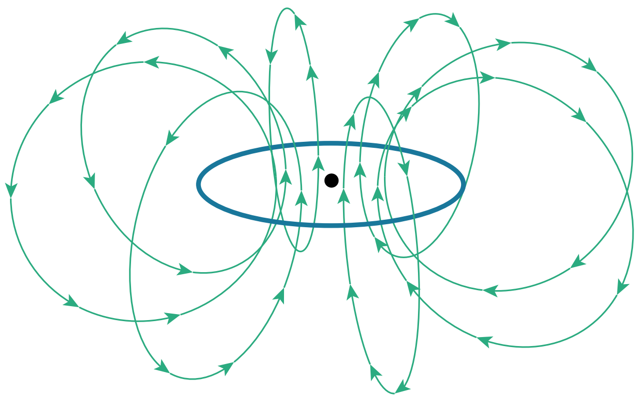
- General screw motion:

$$\mathbf{Q} = Q_{vx}\,\mathbf{e}_{415} + Q_{vy}\,\mathbf{e}_{425} + Q_{vz}\,\mathbf{e}_{435} + Q_{vw}\,\mathbb{1} + Q_{mx}\,\mathbf{e}_{235} + Q_{my}\,\mathbf{e}_{315} + Q_{mz}\,\mathbf{e}_{125} + Q_{mw}\,\mathbf{e}_{5}$$

- General rotoreflection:

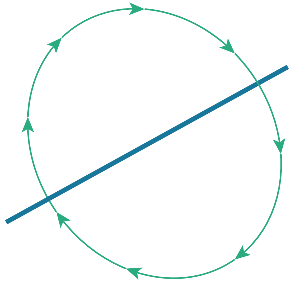
$$\mathbf{F} = F_{px}\,\mathbf{e}_{15} + F_{py}\,\mathbf{e}_{25} + F_{pz}\,\mathbf{e}_{35} + F_{pw}\,\mathbf{e}_{45} + F_{gx}\,\mathbf{e}_{4235} + F_{gy}\,\mathbf{e}_{4315} + F_{gz}\,\mathbf{e}_{4125} + F_{gw}\,\mathbf{e}_{3215}$$

Conformal Motions



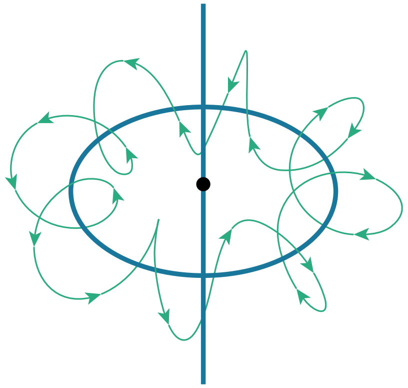
Real Circle / Elliptic Rotation

$$\mathbf{R} = \mathbf{c}\sin\phi + \mathbb{1}\cos\phi$$

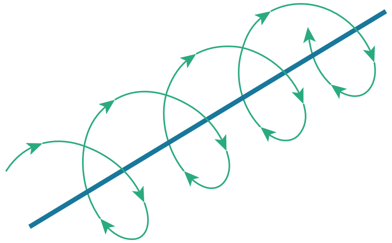


Flat Line / Rotation

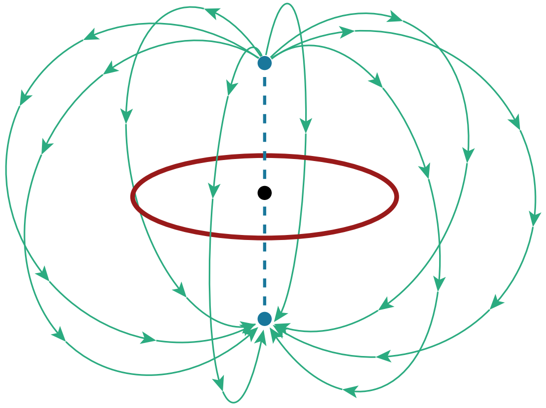
$$\mathbf{R} = \boldsymbol{l}\ \sin\phi + \mathbb{1}\cos\phi$$



Real Circle + Line
Twisted Elliptic Rotation

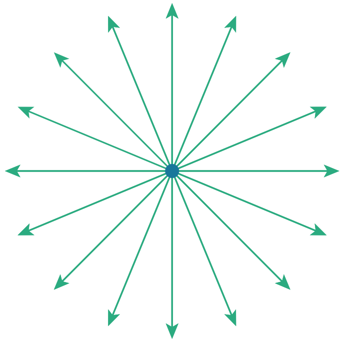


Line or Point in Horizon + Line
Twisted Rotation / Screw Motion



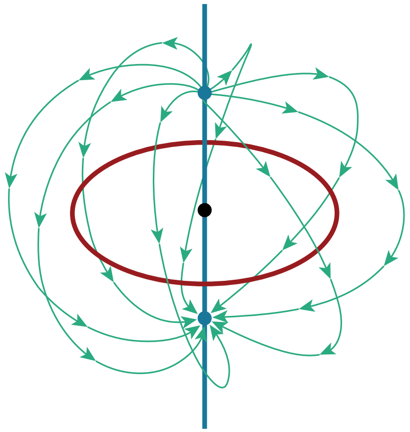
Imaginary Circle / Hyperbolic Rotation

$$\mathbf{R} = \mathbf{c}\sinh\phi + \mathbb{1}\cosh\phi$$

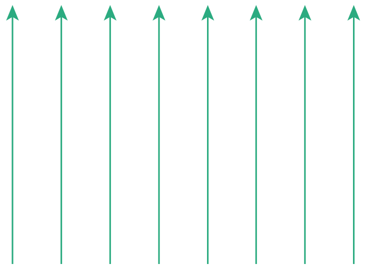


Dual Flat Point / Dilation

$$\mathbf{D} = \frac{1-\sigma}{1+\sigma}\mathbf{p}^{\star} + \mathbb{1}$$



Imaginary Circle + Line
Twisted Hyperbolic Rotation



Line or Point in Horizon /Translation

$$\mathbf{T} = \mathbf{v}^{\star} + \mathbb{1}$$

Conformal Motions

- Operators are equivalent to 5 × 5 matrices
- Simple translation example:

$$\mathbf{T} = \tau_x \mathbf{e}_{235} + \tau_y \mathbf{e}_{315} + \tau_z \mathbf{e}_{125} + \mathbb{1}$$

$$\mathbf{t} = 2\boldsymbol{\tau}$$

$$\begin{bmatrix} 1 & 0 & 0 & t_x & 0 \\ 0 & 1 & 0 & t_y & 0 \\ 0 & 0 & 1 & t_z & 0 \\ 0 & 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & \frac{1}{2}\mathbf{t}^2 & 1 \end{bmatrix}$$

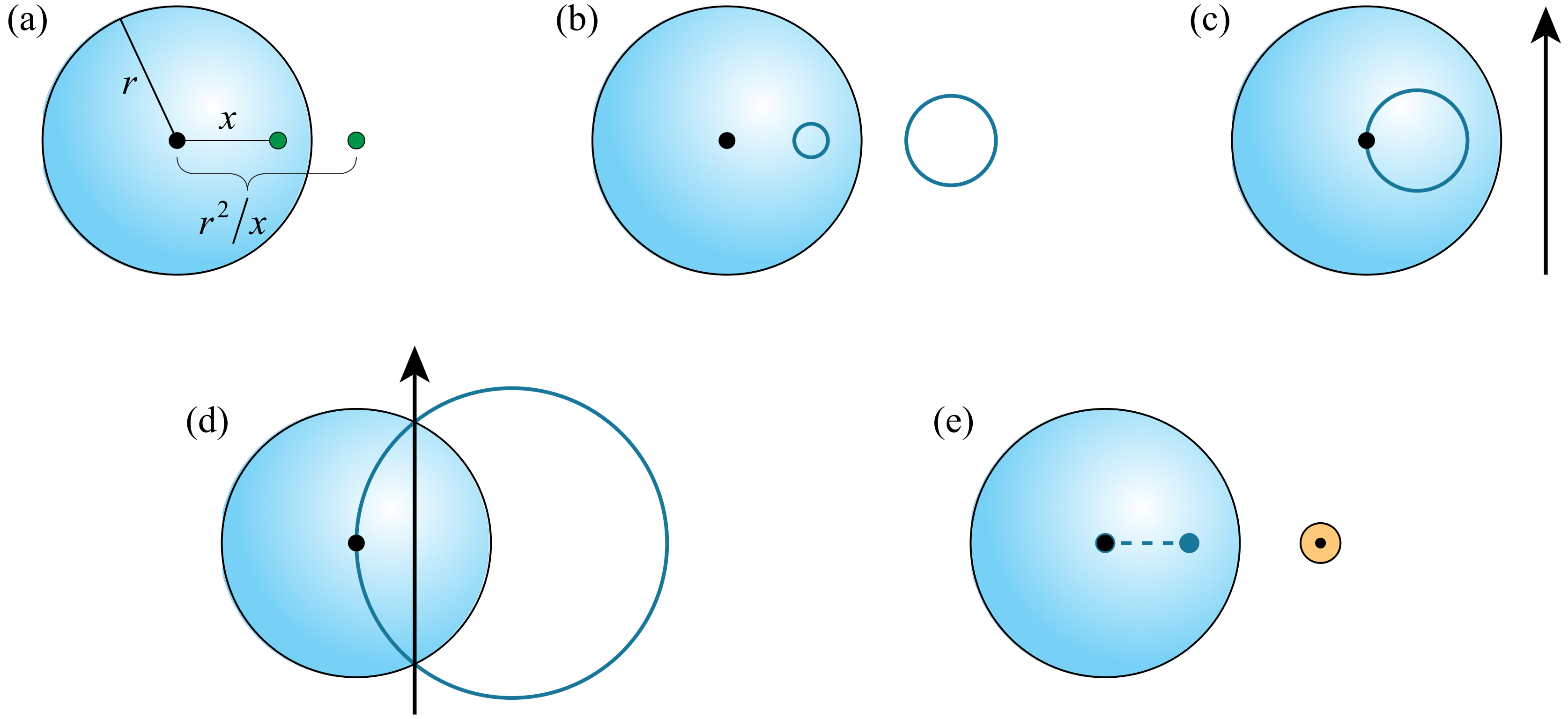
Translation

- Computation gets somewhat absurd
- Would be easier to store object as center, radius, attitude
- Rebuild CGA form as needed

Type	Translation Formula
Flat point **p**	$\mathbf{T} ⟇ \mathbf{p} ⟇ \underset{\sim}{\mathbf{T}} = (p_x + 2\tau_x p_w)\mathbf{e}_{15} + (p_y + 2\tau_y p_w)\mathbf{e}_{25} + (p_z + 2\tau_z p_w)\mathbf{e}_{35} + p_w \mathbf{e}_{45}$
Line ***l***	$\mathbf{T} ⟇ \boldsymbol{l} ⟇ \underset{\sim}{\mathbf{T}} = l_{vx}\mathbf{e}_{415} + l_{vy}\mathbf{e}_{425} + l_{vz}\mathbf{e}_{435} + [l_{mx} + 2(\tau_y l_{vz} - \tau_z l_{vy})]\mathbf{e}_{235}$ $+ [l_{my} + 2(\tau_z l_{vx} - \tau_x l_{vz})]\mathbf{e}_{315} + [l_{mz} + 2(\tau_x l_{vy} - \tau_y l_{vx})]\mathbf{e}_{125}$
Plane **g**	$\mathbf{T} ⟇ \mathbf{g} ⟇ \underset{\sim}{\mathbf{T}} = g_x \mathbf{e}_{4235} + g_y \mathbf{e}_{4315} + g_z \mathbf{e}_{4125} + (g_w - 2\boldsymbol{\tau} \cdot \mathbf{g}_{xyz})\mathbf{e}_{3215}$
Round point **a**	$\mathbf{T} ⟇ \mathbf{a} ⟇ \underset{\sim}{\mathbf{T}} = (a_x + 2\tau_x a_w)\mathbf{e}_1 + (a_y + 2\tau_y a_w)\mathbf{e}_2 + (a_z + 2\tau_z a_w)\mathbf{e}_3 + a_w \mathbf{e}_4$ $+ \left(a_u + 2\boldsymbol{\tau} \cdot \mathbf{a}_{xyz} + 2\boldsymbol{\tau}^2 a_w\right)\mathbf{e}_5$
Dipole **d**	$\mathbf{T} ⟇ \mathbf{d} ⟇ \underset{\sim}{\mathbf{T}} = d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + [d_{mx} + 2(\tau_y d_{vz} - \tau_z d_{vy})]\mathbf{e}_{23}$ $+ [d_{my} + 2(\tau_z d_{vx} - \tau_x d_{vz})]\mathbf{e}_{31} + [d_{mz} + 2(\tau_x d_{vy} - \tau_y d_{vx})]\mathbf{e}_{12}$ $+ [d_{px} + 2(\tau_y d_{mz} - \tau_z d_{my} + \tau_x d_{pw} + 2\tau_x \boldsymbol{\tau} \cdot \mathbf{d}_\mathbf{v} - \boldsymbol{\tau}^2 d_{vx})]\mathbf{e}_{15}$ $+ [d_{py} + 2(\tau_z d_{mx} - \tau_x d_{mz} + \tau_y d_{pw} + 2\tau_y \boldsymbol{\tau} \cdot \mathbf{d}_\mathbf{v} - \boldsymbol{\tau}^2 d_{vy})]\mathbf{e}_{25}$ $+ [d_{pz} + 2(\tau_x d_{my} - \tau_y d_{mx} + \tau_z d_{pw} + 2\tau_z \boldsymbol{\tau} \cdot \mathbf{d}_\mathbf{v} - \boldsymbol{\tau}^2 d_{vz})]\mathbf{e}_{35}$ $+ (d_{pw} + 2\boldsymbol{\tau} \cdot \mathbf{d}_\mathbf{v})\mathbf{e}_{45}$
Circle **c**	$\mathbf{T} ⟇ \mathbf{c} ⟇ \underset{\sim}{\mathbf{T}} = c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412}$ $+ (c_{gw} - 2\boldsymbol{\tau} \cdot \mathbf{c}_{gxyz})\mathbf{e}_{321} + [c_{vx} + 2(\tau_y c_{gz} - \tau_z c_{gy})]\mathbf{e}_{415}$ $+ [c_{vy} + 2(\tau_z c_{gx} - \tau_x c_{gz})]\mathbf{e}_{425} + [c_{vz} + 2(\tau_x c_{gy} - \tau_y c_{gx})]\mathbf{e}_{435}$ $+ [c_{mx} + 2(\tau_y c_{vz} - \tau_z c_{vy} - \tau_x c_{gw} + 2\tau_x \boldsymbol{\tau} \cdot \mathbf{c}_{gxyz} - \boldsymbol{\tau}^2 c_{gx})]\mathbf{e}_{235}$ $+ [c_{my} + 2(\tau_z c_{vx} - \tau_x c_{vz} - \tau_y c_{gw} + 2\tau_y \boldsymbol{\tau} \cdot \mathbf{c}_{gxyz} - \boldsymbol{\tau}^2 c_{gy})]\mathbf{e}_{315}$ $+ [c_{mz} + 2(\tau_x c_{vy} - \tau_y c_{vx} - \tau_z c_{gw} + 2\tau_z \boldsymbol{\tau} \cdot \mathbf{c}_{gxyz} - \boldsymbol{\tau}^2 c_{gz})]\mathbf{e}_{125}$
Sphere **s**	$\mathbf{T} ⟇ \mathbf{s} ⟇ \underset{\sim}{\mathbf{T}} = s_u \mathbf{e}_{1234} + (s_x - 2\tau_x s_u)\mathbf{e}_{4235} + (s_y - 2\tau_y s_u)\mathbf{e}_{4315} + (s_z - 2\tau_z s_u)\mathbf{e}_{4125}$ $+ \left(s_w - 2\boldsymbol{\tau} \cdot \mathbf{s}_{xyz} + 2\boldsymbol{\tau}^2 s_u\right)\mathbf{e}_{3215}$

Sphere Inversion

- In CGA, reflections across planes generalize to reflections through spheres



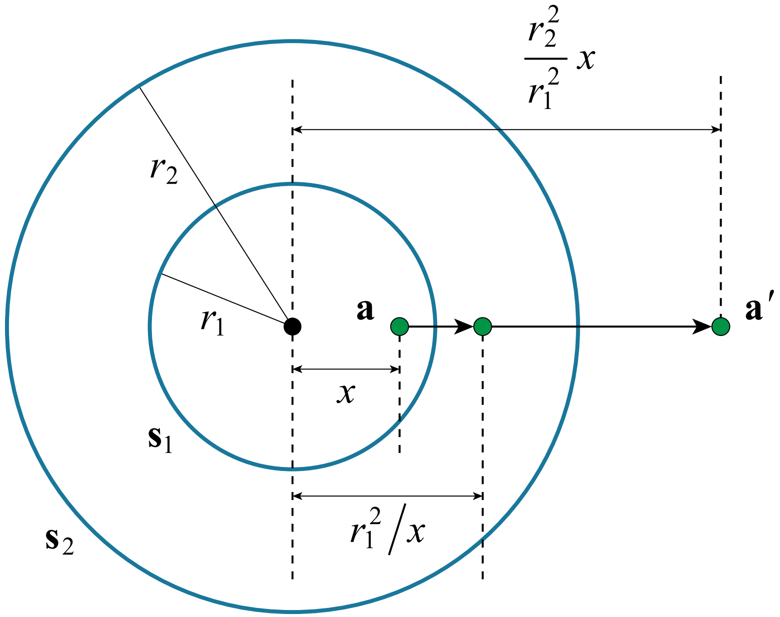
Sphere Inversion

- For sphere of radius *r* centered at $\left(m_x, m_y, m_z\right)$, points are transformed by

$$\begin{bmatrix} r^2 - 2m_x^2 & -2m_x m_y & -2m_x m_z & \left(\mathbf{m}^2 - r^2\right) m_x & 2m_x \\ -2m_x m_y & r^2 - 2m_y^2 & -2m_y m_z & \left(\mathbf{m}^2 - r^2\right) m_y & 2m_y \\ -2m_x m_z & -2m_y m_z & r^2 - 2m_z^2 & \left(\mathbf{m}^2 - r^2\right) m_z & 2m_z \\ -2m_x & -2m_y & -2m_z & \mathbf{m}^2 & 2 \\ -\left(\mathbf{m}^2 - r^2\right) m_x & -\left(\mathbf{m}^2 - r^2\right) m_y & -\left(\mathbf{m}^2 - r^2\right) m_z & \frac{1}{2}\left(\mathbf{m}^2 - r^2\right)^2 & \mathbf{m}^2 \end{bmatrix}$$

Dilation

- Translation results from reflections across two parallel planes
- This generalizes to reflections through two concentric spheres
- Result is a dilation about the center of the spheres



Dilation

- Operator that dilates by factor σ about center $\left(m_x, m_y, m_z\right)$

$$\mathbf{D} = \frac{1-\sigma}{2}\left(m_x \mathbf{e}_{235} + m_y \mathbf{e}_{315} + m_z \mathbf{e}_{125} - \mathbf{e}_{321}\right) + \frac{1+\sigma}{2}\mathbb{1}$$

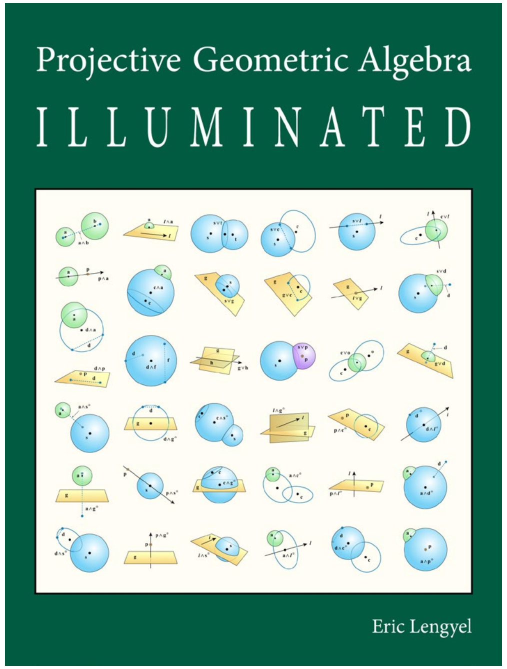
$$\begin{bmatrix} \sigma & 0 & 0 & \left(1-\sigma\right)m_x & 0 \\ 0 & \sigma & 0 & \left(1-\sigma\right)m_y & 0 \\ 0 & 0 & \sigma & \left(1-\sigma\right)m_z & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \sigma\left(1-\sigma\right)m_x & \sigma\left(1-\sigma\right)m_y & \sigma\left(1-\sigma\right)m_z & \tfrac{1}{2}\left(1-\sigma\right)^2\mathbf{m}^2 & \sigma^2 \end{bmatrix}$$

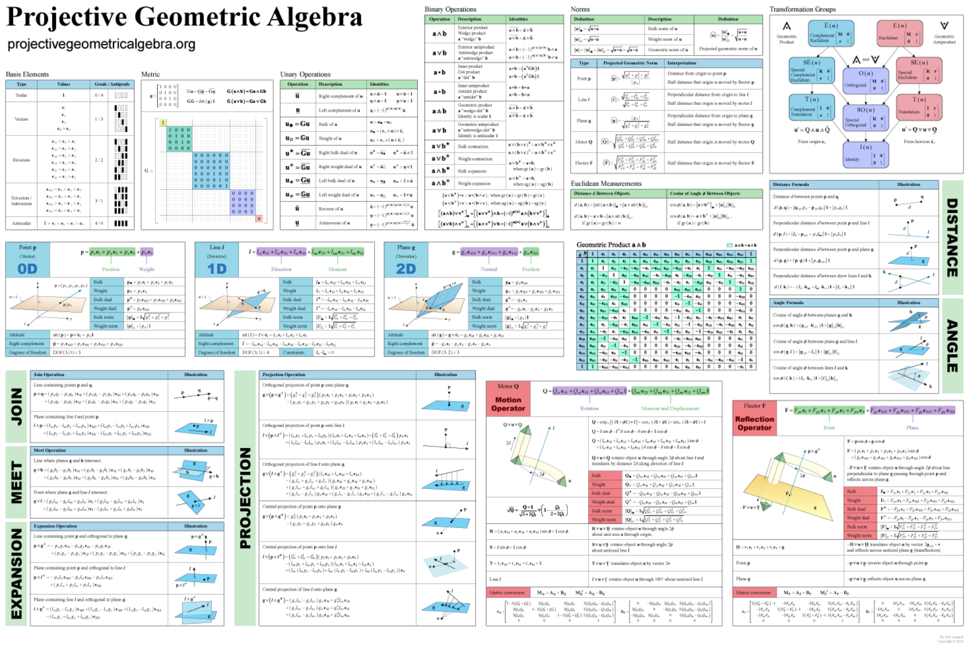
Dilation

Type	Dilation Formula
Flat point **p**	$\mathbf{D} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{D}} = \left[\sigma^2 p_x + \sigma(1-\sigma) m_x p_w\right]\mathbf{e}_{15} + \left[\sigma^2 p_y + \sigma(1-\sigma) m_y p_w\right]\mathbf{e}_{25}$ $+\left[\sigma^2 p_z + \sigma(1-\sigma) m_z p_w\right]\mathbf{e}_{35} + \sigma p_w \mathbf{e}_{45}$
Line ***l***	$\mathbf{D} \vee \boldsymbol{l} \vee \underset{\sim}{\mathbf{D}} = \sigma l_{vx}\mathbf{e}_{415} + \sigma l_{vy}\mathbf{e}_{425} + \sigma l_{vz}\mathbf{e}_{435}$ $+\left[\sigma^2 l_{mx} + \sigma(1-\sigma)\left(m_y l_{vz} - m_z l_{vy}\right)\right]\mathbf{e}_{235}$ $+\left[\sigma^2 l_{my} + \sigma(1-\sigma)\left(m_z l_{vx} - m_x l_{vz}\right)\right]\mathbf{e}_{315}$ $+\left[\sigma^2 l_{mz} + \sigma(1-\sigma)\left(m_x l_{vy} - m_y l_{vx}\right)\right]\mathbf{e}_{125}$
Plane **g**	$\mathbf{D} \vee \mathbf{g} \vee \underset{\sim}{\mathbf{D}} = \sigma g_x \mathbf{e}_{4235} + \sigma g_y \mathbf{e}_{4315} + \sigma g_z \mathbf{e}_{4125}$ $+\left[\sigma^2 g_w - \sigma(1-\sigma)\mathbf{m}\cdot\mathbf{g}_{xyz}\right]\mathbf{e}_{3215}$
Round point **a**	$\mathbf{D} \vee \mathbf{a} \vee \underset{\sim}{\mathbf{D}} = \left(\sigma a_x + (1-\sigma) m_x a_w\right)\mathbf{e}_1 + \left(\sigma a_y + (1-\sigma) m_y a_w\right)\mathbf{e}_2$ $+\left(\sigma a_z + (1-\sigma) m_z a_w\right)\mathbf{e}_3 + a_w \mathbf{e}_4$ $+\left[\sigma^2 a_u + \sigma(1-\sigma)\mathbf{m}\cdot\mathbf{a}_{xyz} + \tfrac{1}{2}(1-\sigma)^2\,\mathbf{m}^2 a_w\right]\mathbf{e}_5$
Dipole **d**	$\mathbf{D} \vee \mathbf{d} \vee \underset{\sim}{\mathbf{D}} = d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + \left[\sigma d_{mx} + (1-\sigma)\left(m_y d_{vz} - m_z d_{vy}\right)\right]\mathbf{e}_{23}$ $+\left[\sigma d_{my} + (1-\sigma)\left(m_z d_{vx} - m_x d_{vz}\right)\right]\mathbf{e}_{31} + \left[\sigma d_{mz} + (1-\sigma)\left(m_x d_{vy} - m_y d_{vx}\right)\right]\mathbf{e}_{12}$ $+\left[\sigma^2 d_{px} + \sigma(1-\sigma)\left(m_y d_{mz} - m_z d_{my} + m_x d_{pw}\right) + \tfrac{1}{2}(1-\sigma)^2\left(2m_x\mathbf{m}\cdot\mathbf{d}_\mathbf{v} - \mathbf{m}^2 d_{vx}\right)\right]\mathbf{e}_{15}$ $+\left[\sigma^2 d_{py} + \sigma(1-\sigma)\left(m_z d_{mx} - m_x d_{mz} + m_y d_{pw}\right) + \tfrac{1}{2}(1-\sigma)^2\left(2m_y\mathbf{m}\cdot\mathbf{d}_\mathbf{v} - \mathbf{m}^2 d_{vy}\right)\right]\mathbf{e}_{25}$ $+\left[\sigma^2 d_{py} + \sigma(1-\sigma)\left(m_x d_{my} - m_y d_{mx} + m_z d_{pw}\right) + \tfrac{1}{2}(1-\sigma)^2\left(2m_z\mathbf{m}\cdot\mathbf{d}_\mathbf{v} - \mathbf{m}^2 d_{vz}\right)\right]\mathbf{e}_{35}$ $+\left[\sigma d_{pw} + (1-\sigma)\mathbf{m}\cdot\mathbf{d}_\mathbf{v}\right]\mathbf{e}_{45}$
Circle **c**	$\mathbf{D} \vee \mathbf{c} \vee \underset{\sim}{\mathbf{D}} = c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412}$ $+\left[\sigma c_{gw} - (1-\sigma)\mathbf{m}\cdot\mathbf{c}_{gxyz}\right]\mathbf{e}_{321} + \left[\sigma c_{vx} + (1-\sigma)\left(m_y c_{gz} - m_z c_{gy}\right)\right]\mathbf{e}_{415}$ $+\left[\sigma c_{vy} + (1-\sigma)\left(m_z c_{gx} - m_x c_{gz}\right)\right]\mathbf{e}_{425} + \left[\sigma c_{vz} + (1-\sigma)\left(m_x c_{gy} - m_y c_{gx}\right)\right]\mathbf{e}_{435}$ $+\left[\sigma^2 c_{mx} + \sigma(1-\sigma)\left(m_y c_{vz} - m_z c_{vy} - m_x c_{gw}\right) + \tfrac{1}{2}(1-\sigma)^2\left(2m_x\mathbf{m}\cdot\mathbf{c}_{gxyz} - \mathbf{m}^2 c_{gx}\right)\right]\mathbf{e}_{235}$ $+\left[\sigma^2 c_{my} + \sigma(1-\sigma)\left(m_z c_{vx} - m_x c_{vz} - m_y c_{gw}\right) + \tfrac{1}{2}(1-\sigma)^2\left(2m_y\mathbf{m}\cdot\mathbf{c}_{gxyz} - \mathbf{m}^2 c_{gy}\right)\right]\mathbf{e}_{315}$ $+\left[\sigma^2 c_{mz} + \sigma(1-\sigma)\left(m_x c_{vy} - m_y c_{vx} - m_z c_{gw}\right) + \tfrac{1}{2}(1-\sigma)^2\left(2m_z\mathbf{m}\cdot\mathbf{c}_{gxyz} - \mathbf{m}^2 c_{gz}\right)\right]\mathbf{e}_{125}$
Sphere **s**	$\mathbf{D} \vee \mathbf{s} \vee \underset{\sim}{\mathbf{D}} = s_u\mathbf{e}_{1234} + \left(\sigma s_x - (1-\sigma) m_x s_u\right)\mathbf{e}_{4235}$ $+\left(\sigma s_y - (1-\sigma) m_y s_u\right)\mathbf{e}_{4315} + \left(\sigma s_z - (1-\sigma) m_z s_u\right)\mathbf{e}_{4125}$ $+\left[\sigma^2 s_w - \sigma(1-\sigma)\mathbf{m}\cdot\mathbf{s}_{xyz} + \tfrac{1}{2}(1-\sigma)^2\,\mathbf{m}^2 s_u\right]\mathbf{e}_{3215}$

References

- Projective Geometric Algebra Illuminated
- projectivegeometricalgebra.org

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