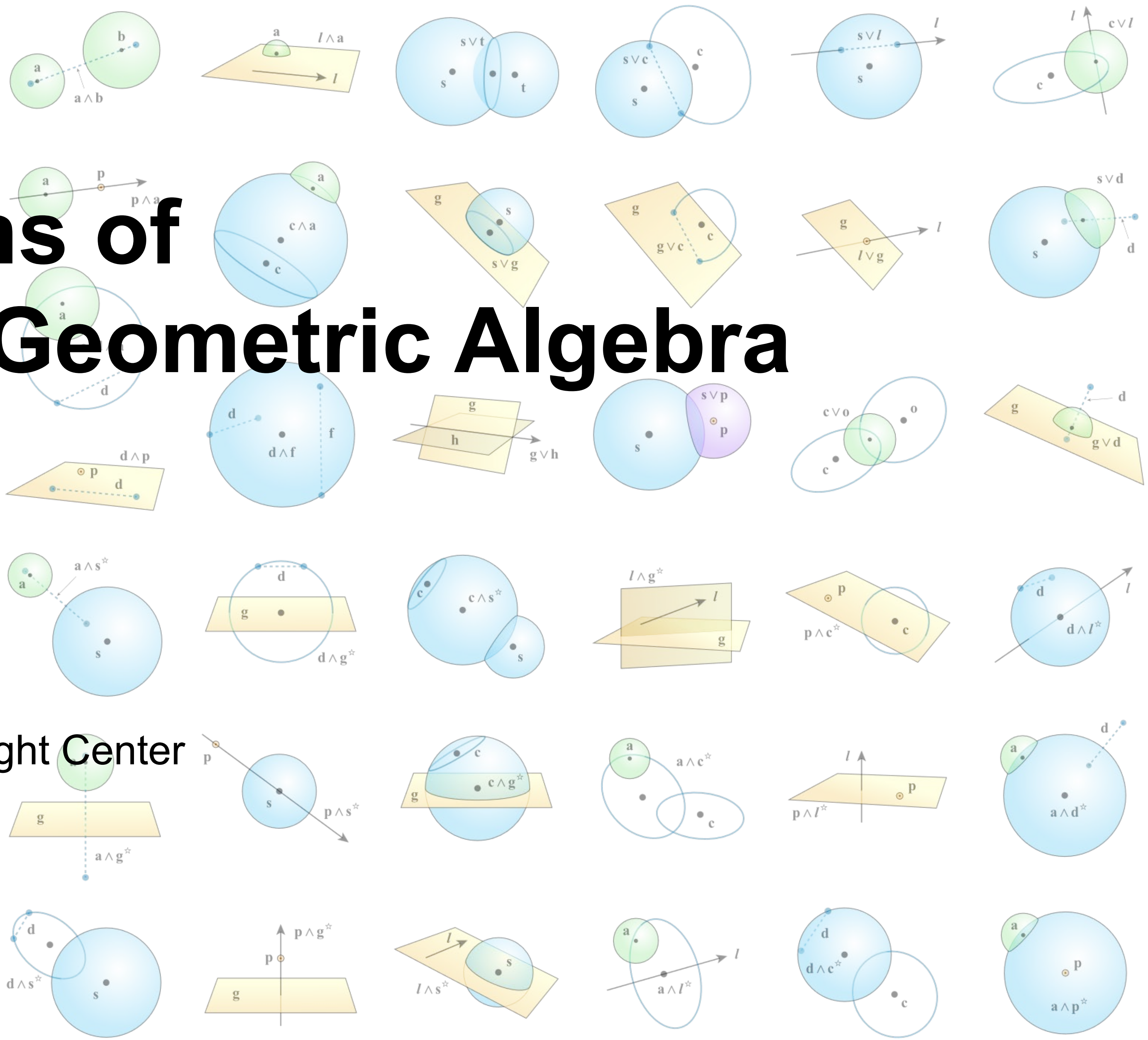
Foundations of
Projective Geometric Algebra



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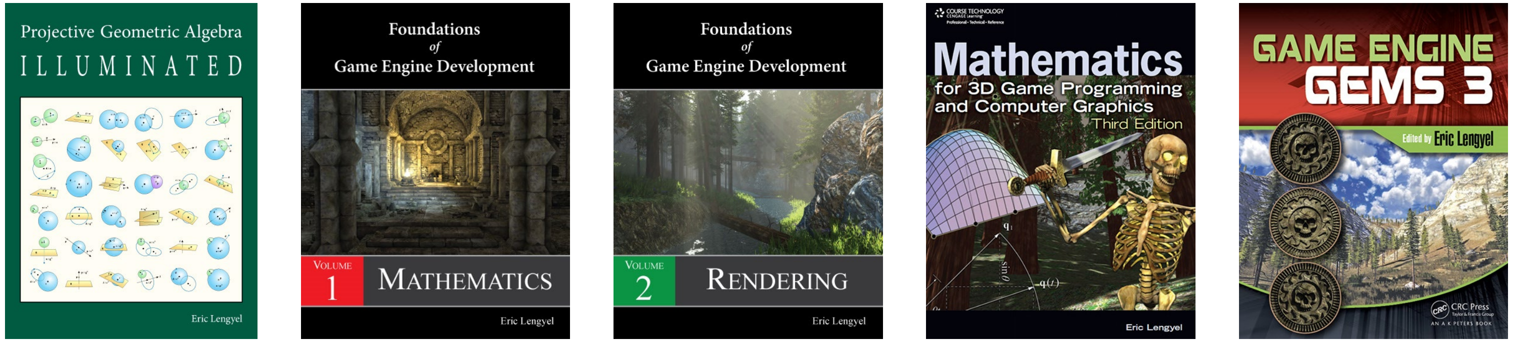
About the Speaker

- Computer Scientist / Mathematician
- Working in industry since 1994
- Running company that specializes in digital typography and game engines
- Writing books



SLUG

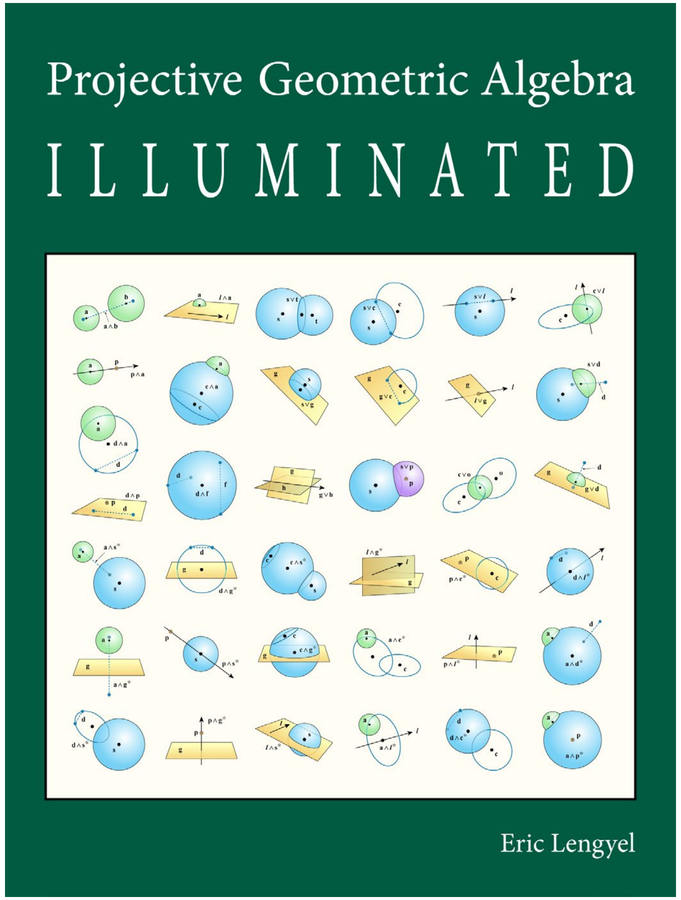
C4 ENGINE

OpenGEX

TRANSVOXEL

Subject of This Talk

- 4D rigid exterior algebra
  - Homogeneous representation of 3D geometry
  - Points, lines, planes
  - Join, meet, projection, norm, distance, angle
- 4D rigid geometric algebra
  - Euclidean isometries in 3D space
  - Rotations, translations, screw transformations
- 5D conformal exterior algebra (briefly)
  - Round points, dipoles, circles, spheres
- Details in PGA Illuminated



Exterior / Grassmann Algebra

- Wedge product $\wedge$
  - Combines dimensions of operands
  - Vectors square to zero:

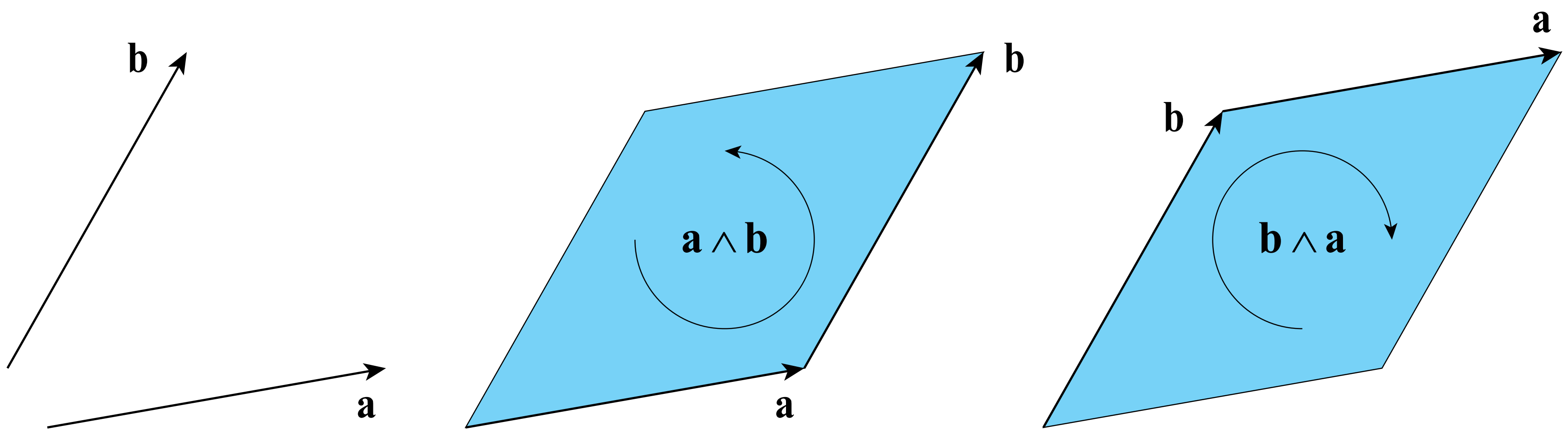
$$\mathbf{v} \wedge \mathbf{v} = 0$$

  - Antisymmetric on vectors:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

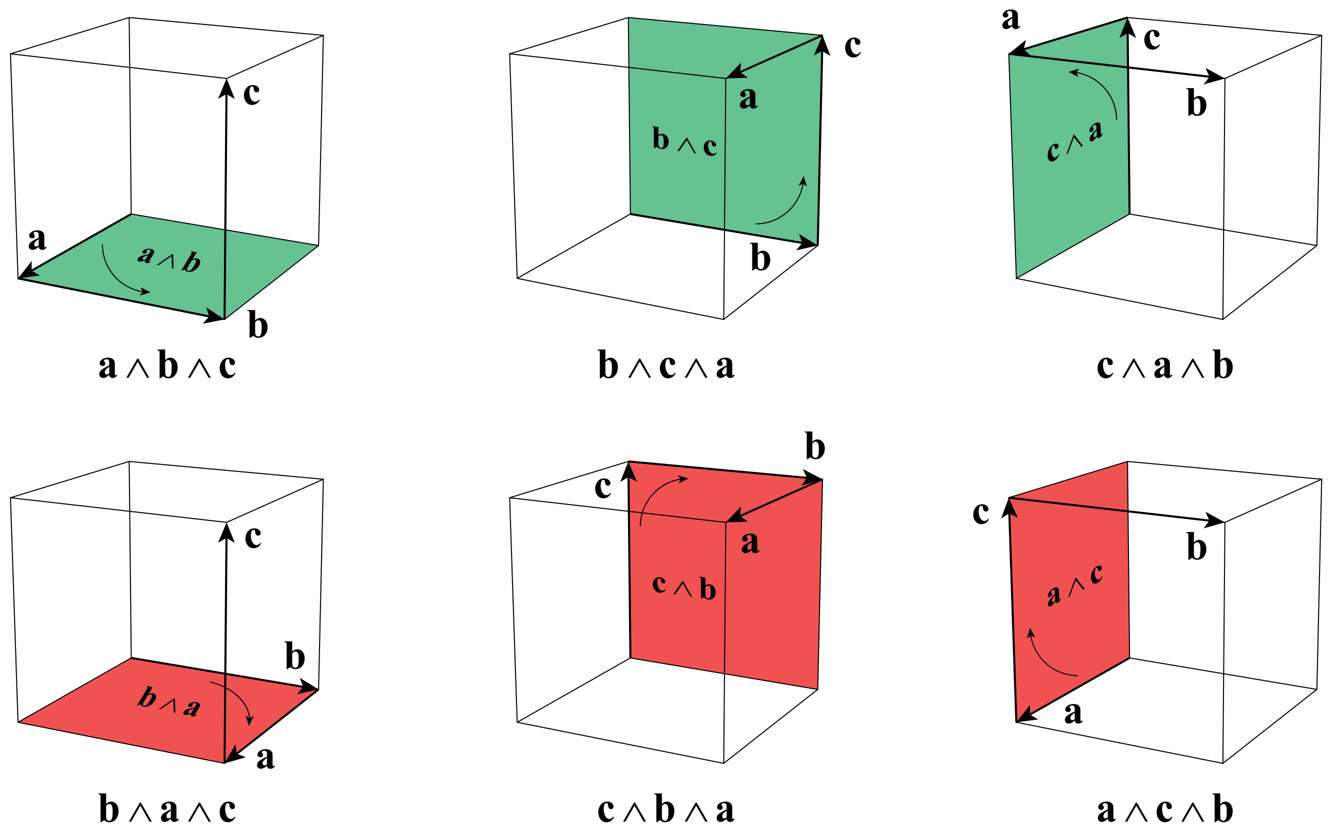
Bivectors

- Wedge product of two vectors **a** and **b**

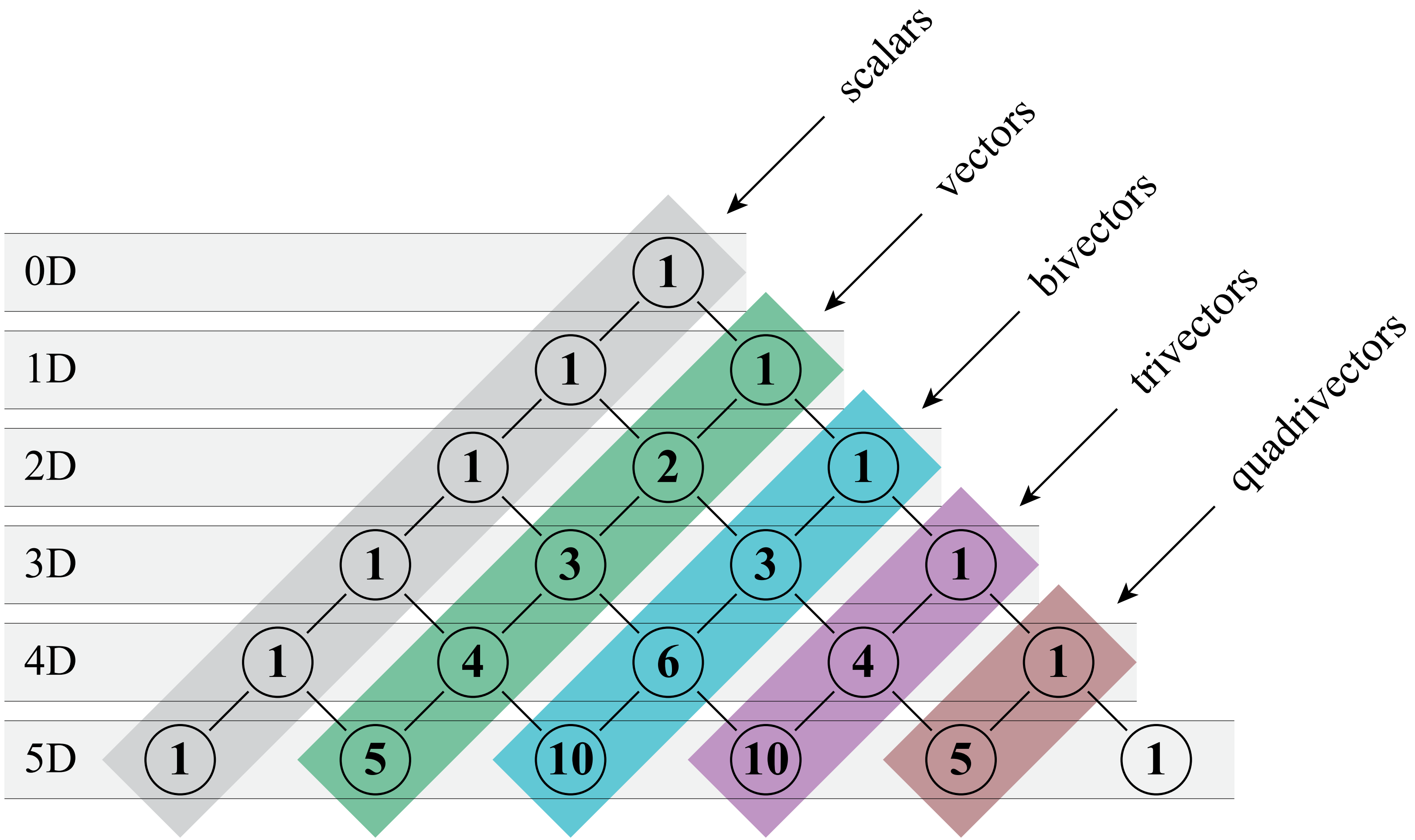


Trivectors

- Wedge product of three vectors **a**, **b**, and **c**



Pascal's Triangle



Rigid Exterior / Geometric Algebra

- Projective algebra with one extra dimension
- Contains points, lines, planes in 3D
- Can perform rotations, translations, screw transformations

4D Exterior Algebra

- Extends 4D vector space

- One scalar **1**
- Four vector basis elements
- Six bivector basis elements
- Four trivector basis elements
- One antiscalar $\mathbb{1}$

Type	Values	Grade / Antigrade	
Scalar	$\mathbf{1}$	0 / 4	▯▯▯▯
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4 = \mathbf{e}_n$	1 / 3	▮▯▯▯ ▯▮▯▯ ▯▯▮▯ ▯▯▯▮
Bivectors	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$	2 / 2	▮▯▯▮ ▯▮▯▮ ▯▯▮▮ ▯▮▮▯ ▮▯▮▯ ▮▮▯▯
Trivectors / Antivectors	$\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$	3 / 1	▯▮▮▮ ▮▯▮▮ ▮▮▯▮ ▮▮▮▯
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	▮▮▮▮

4D Exterior Product

Wedge Product $\mathbf{a} \wedge \mathbf{b}$

$\mathbf{a}$ \ $\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_{1}$	$\mathbf{e}_{1}$	0	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	0	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0	0
$\mathbf{e}_{2}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	0	$-\mathbf{e}_{423}$	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0
$\mathbf{e}_{3}$	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	0	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	0	0	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0
$\mathbf{e}_{4}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	0	$\mathbf{e}_{423}$	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	0	0	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	0	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}$	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Complements

- Complement inverts full / empty dimensions
- Right complement denoted by overbar
- Left complement denoted by underbar
- For basis element **u**,

$$\mathbf{u} \wedge \overline{\mathbf{u}} = \mathbb{1} \qquad \underline{\mathbf{u}} \wedge \mathbf{u} = \mathbb{1}$$

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\overline{\mathbf{u}}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$\mathbf{1}$
$\underline{\mathbf{u}}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{1}$

Antiproducts

- Antiwedge product denoted by $\vee$
- Wedge product combines dimensions that are *present*
  - Adds grades
- Antiwedge product combines dimensions that are *absent*
  - Adds antigrades

De Morgan Laws

- Every operation with “anti” in name satisfies a De Morgan law:

$$\overline{\mathbf{a} \vee \mathbf{b}} = \overline{\mathbf{a}} \wedge \overline{\mathbf{b}} \qquad \underline{\mathbf{a} \vee \mathbf{b}} = \underline{\mathbf{a}} \wedge \underline{\mathbf{b}}$$

- To calculate anti-operation,
  - Take a complement of each input
  - Perform the regular operation
  - Take opposite complement of the result

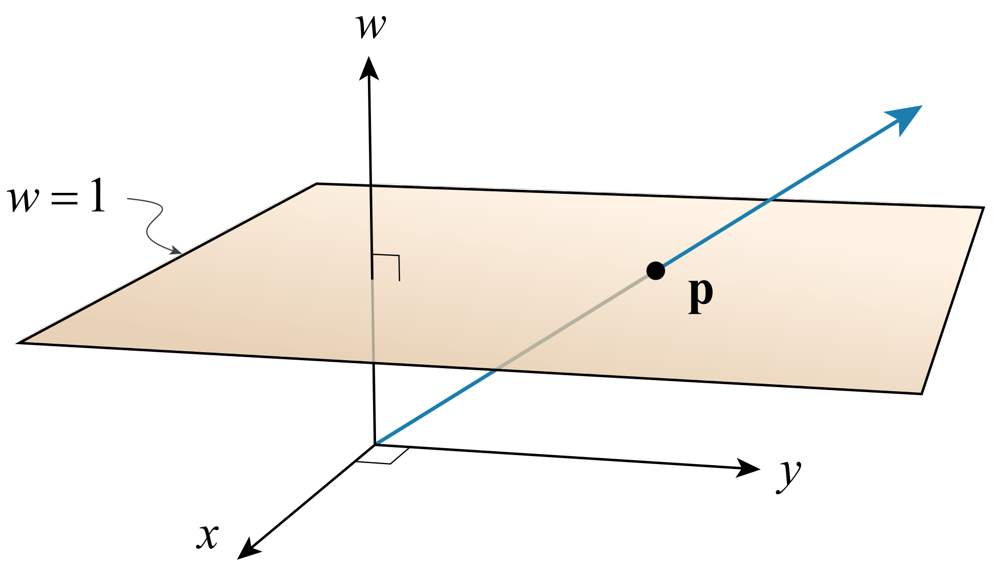
4D Exterior Antiproduct

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	$\mathbf{e}_3$
$\mathbf{e}_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	$\mathbf{e}_4$
$\mathbf{e}_{41}$	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	0	0	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	0	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	0	0	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	0	$-\mathbf{1}$	0	0	0	0	0	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	0	0	$-\mathbf{1}$	0	0	0	0	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	0	0	0	$-\mathbf{e}_3$	$\mathbf{e}_2$	0	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	0	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_3$	0	$-\mathbf{e}_1$	$\mathbf{e}_{43}$	0	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	0	0	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$-\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Point

$$\mathbf{p} = \underbrace{p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3}_{\text{Position}} + \underbrace{p_w \mathbf{e}_4}_{\text{Weight}}$$



Special Points

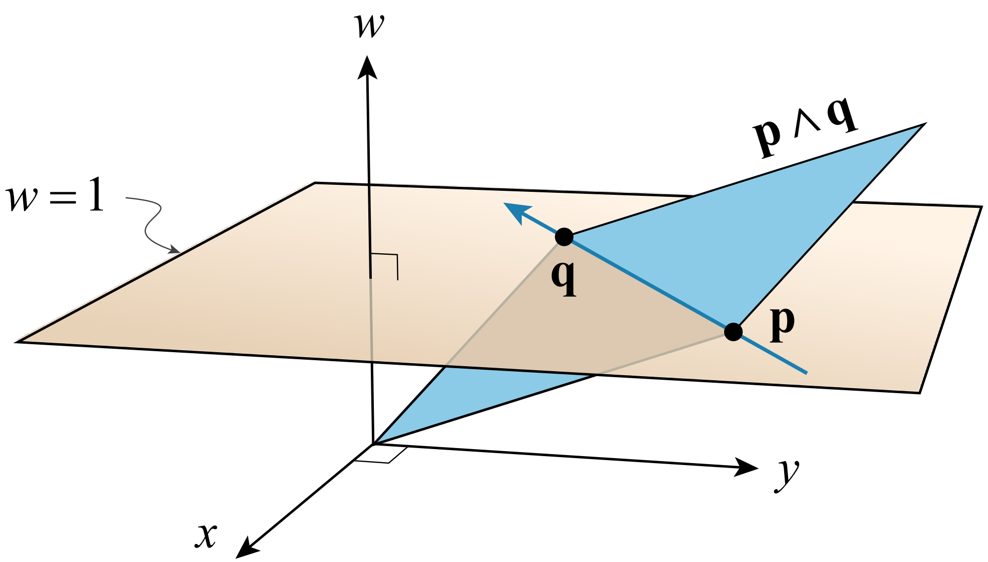
- The origin is simply the point $\mathbf{e}_4$
- Point with zero weight lies at infinity in (x, y, z) direction
- Points at infinity in opposite directions are equivalent

Line

$$\mathbf{p} \wedge \mathbf{q} = (q_x p_w - p_x q_w)\,\mathbf{e}_{41} + (q_y p_w - p_y q_w)\,\mathbf{e}_{42} + (q_z p_w - p_z q_w)\,\mathbf{e}_{43}$$
$$+ (p_y q_z - p_z q_y)\,\mathbf{e}_{23} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}$$

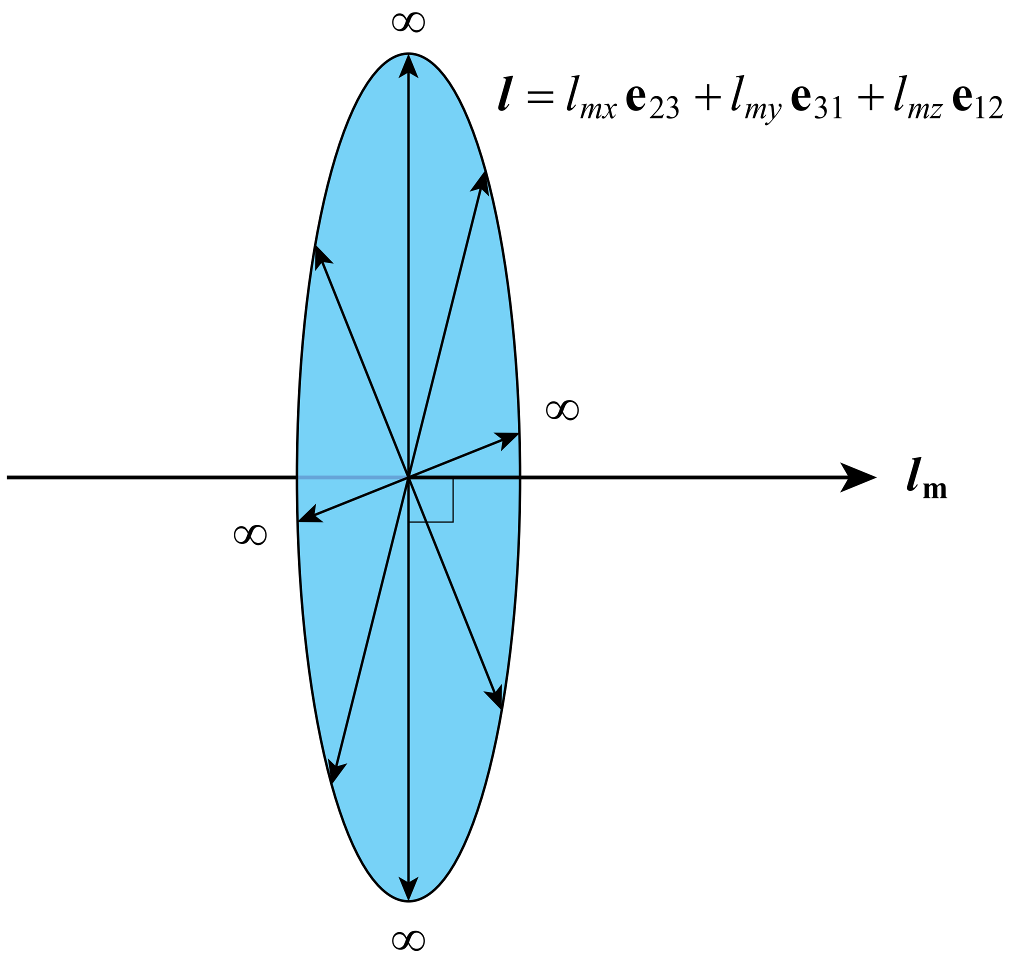
$$\boldsymbol{l} = \underbrace{l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}}_{\text{Moment}}$$

$$\boldsymbol{l}_{\mathbf{v}} \cdot \boldsymbol{l}_{\mathbf{m}} = 0$$



Lines at Infinity

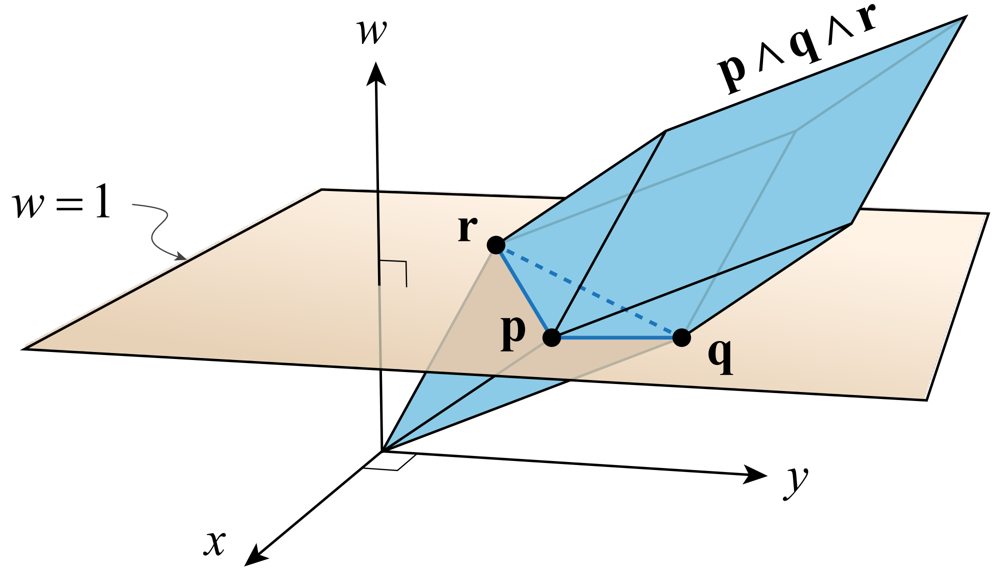
- Line with zero direction lies at infinity



Plane

$$\boldsymbol{l} \wedge \mathbf{p} = (l_{vy}p_z - l_{vz}p_y + l_{mx})\,\overline{\mathbf{e}}_1 + (l_{vz}p_x - l_{vx}p_z + l_{my})\,\overline{\mathbf{e}}_2 + (l_{vx}p_y - l_{vy}p_x + l_{mz})\,\overline{\mathbf{e}}_3 - (l_{mx}p_x + l_{my}p_y + l_{mz}p_z)\,\overline{\mathbf{e}}_4$$

$$\mathbf{g} = \underbrace{g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w\mathbf{e}_{321}}_{\text{Position}}$$

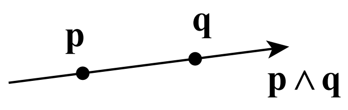
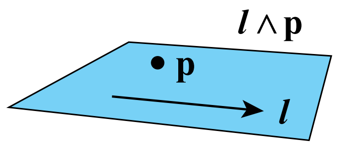


Horizon

- Plane with zero normal lies at infinity $g_w \mathbf{e}_{321}$
- Contains all points at infinity, all lines at infinity
- Given special name *horizon*
- Complement of origin

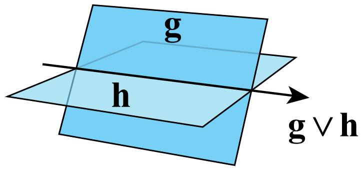
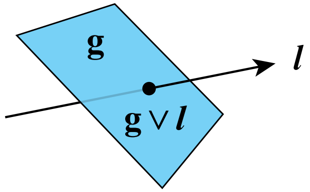
Join

- Wedge product performs join operation

Join Operation	**Illustration**
Line containing points **p** and **q**. $$\mathbf{p}\wedge\mathbf{q}=(p_w q_x - p_x q_w)\,\mathbf{e}_{41}+(p_w q_y - p_y q_w)\,\mathbf{e}_{42}+(p_w q_z - p_z q_w)\,\mathbf{e}_{43} +(p_y q_z - p_z q_y)\,\mathbf{e}_{23}+(p_z q_x - p_x q_z)\,\mathbf{e}_{31}+(p_x q_y - p_y q_x)\,\mathbf{e}_{12}$$	
Plane containing line $\boldsymbol{l}$ and point **p**. $$\boldsymbol{l}\wedge\mathbf{p}=(l_{vy}p_z - l_{vz}p_y + l_{mx}p_w)\,\mathbf{e}_{423}+(l_{vz}p_x - l_{vx}p_z + l_{my}p_w)\,\mathbf{e}_{431} +(l_{vx}p_y - l_{vy}p_x + l_{mz}p_w)\,\mathbf{e}_{412}-(l_{mx}p_x + l_{my}p_y + l_{mz}p_z)\,\mathbf{e}_{321}$$	

Meet

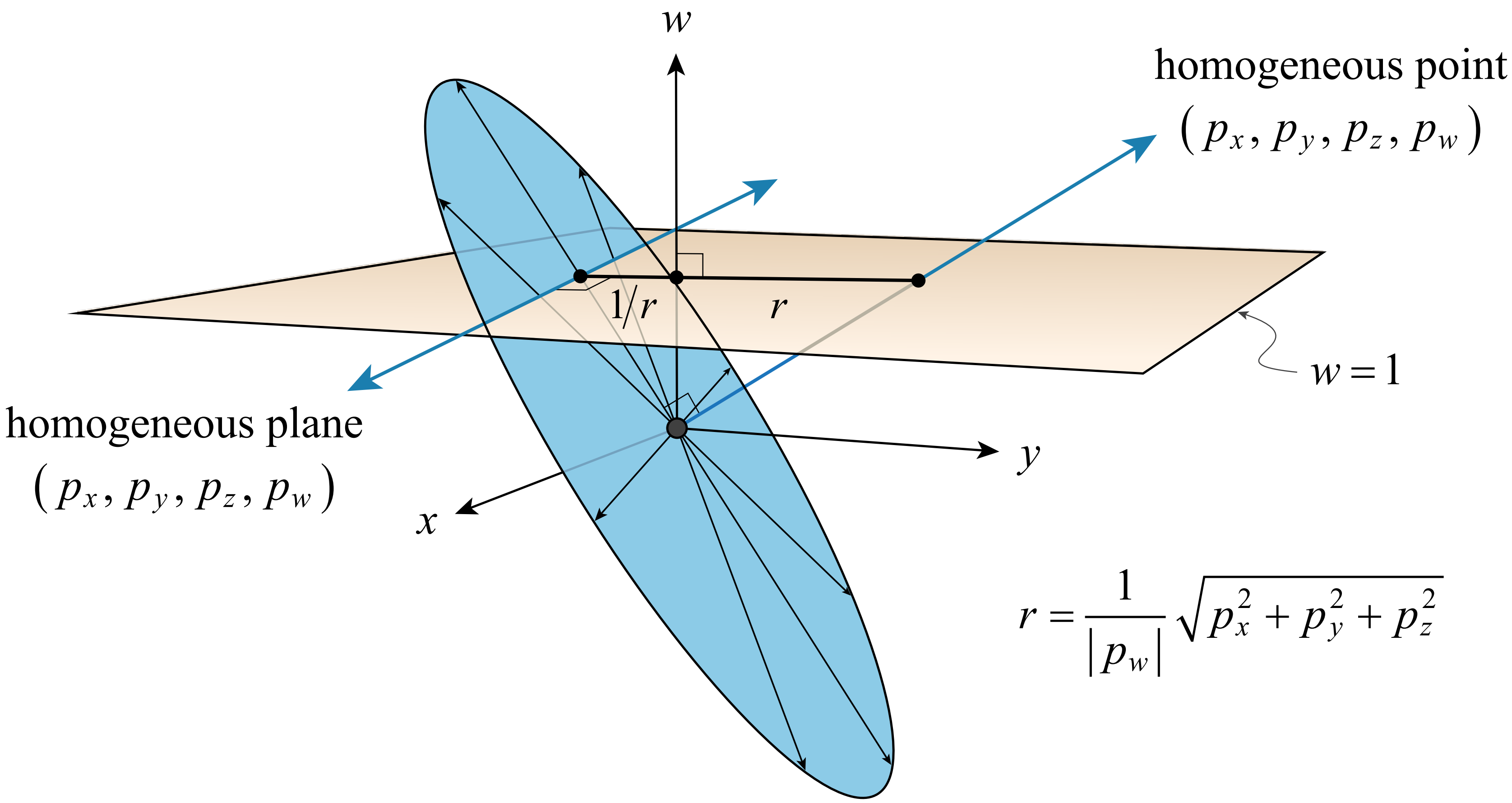
- Antiwedge product performs meet operation

Meet Operation	**Illustration**
Line where planes **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\,\mathbf{e}_{41} + (g_x h_z - g_z h_x)\,\mathbf{e}_{42} + (g_y h_x - g_x h_y)\,\mathbf{e}_{43}$ $\quad + (g_x h_w - g_w h_x)\,\mathbf{e}_{23} + (g_y h_w - g_w h_y)\,\mathbf{e}_{31} + (g_z h_w - g_w h_z)\,\mathbf{e}_{12}$	
Point where plane **g** and line ***l*** intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\,\mathbf{e}_1 + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\,\mathbf{e}_2$ $\quad + (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\,\mathbf{e}_3 - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\,\mathbf{e}_4$	

Duality

- Every object can be interpreted as two different things
- Every operation performs two different actions
- One interpretation corresponds to regular space
- The other interpretation corresponds to *antispace*

Duality



w
homogeneous point
(p_x, p_y, p_z, p_w)
$1/r$
r
$w = 1$
homogeneous plane
(p_x, p_y, p_z, p_w)
x
y
$$r = \frac{1}{|p_w|}\sqrt{p_x^2 + p_y^2 + p_z^2}$$

Exomorphisms

- Given an n x n linear transformation $\mathbf{m}$ that operates on vectors
- The exomorphism $\mathbf{M}$ is the 2^n x 2^n matrix that operates on the whole algebra
- Exomorphism preserves structure under the wedge product:

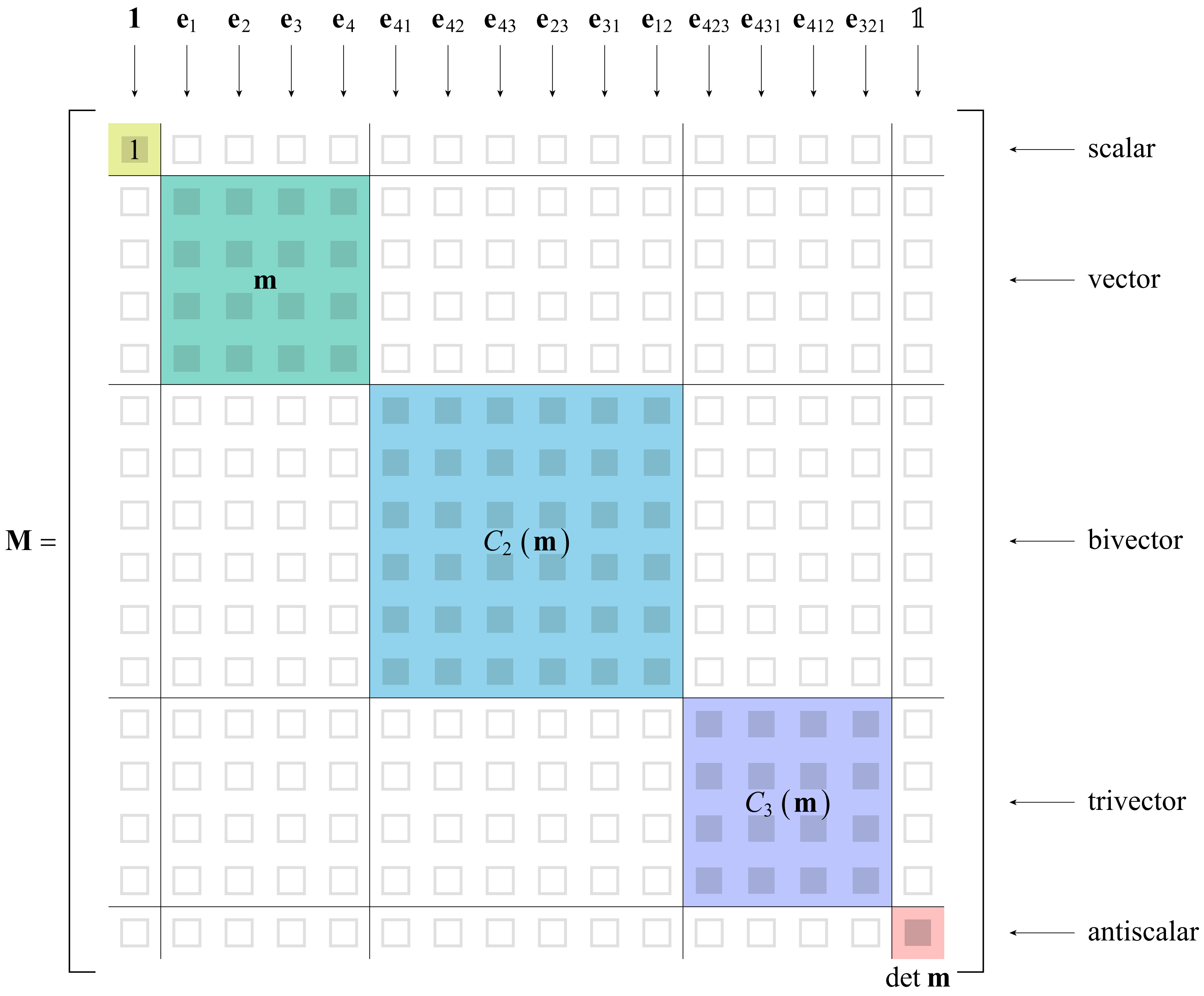
$$\mathbf{M}(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{Ma}) \wedge (\mathbf{Mb})$$

Exomorphisms

- Matrix **M** is block diagonal

- Each block has columns given by wedge products of columns of the original matrix **m**

- These are called *compound matrices* of **m**


1 e1 e2 e3 e4 e41 e42 e43 e23 e31 e12 e423 e431 e412 e321 𝟙
M =
1
m
C2 (m)
C3 (m)
det m
scalar
vector
bivector
trivector
antiscalar

Translation Exomorphism

$$\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -t_z & t_y & 1 & 0 & 0 \\ t_z & 0 & -t_x & 0 & 1 & 0 \\ -t_y & t_x & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_x & -t_y & -t_z & 1 \end{bmatrix}$$

The Metric Tensor

- *n* x *n* matrix that defines dot products of vectors

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$

$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_4 \cdot \mathbf{e}_4 = 0$$

$$\mathfrak{g}_{ij} \equiv \mathbf{v}_i \cdot \mathbf{v}_j$$

Metric Exomorphism

- The metric tensor is a linear transformation
- Thus, it can be extended to a full exomorphism matrix **G**
- There is also a metric *antiexomorphism*, or just “antimetric”, that satisfies

$$\mathbb{G}\mathbf{u} = \underline{\mathbf{G}\overline{\mathbf{u}}} = \overline{\mathbf{G}\underline{\mathbf{u}}}$$

Metric and Antimetric

$$\mathbf{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0
\end{array}\right]$$

$$\mathbb{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1
\end{array}\right]$$

$$\mathbf{G}\mathbb{G} = \det(\mathfrak{g})\,\mathbf{I}$$

Bulk and Weight

- Multiplying 2^n-dimensional multivector by metric or antimetric partitions into two pieces

- Bulk $\mathbf{u}_\bullet = \mathbf{G}\mathbf{u}$ All components without factor $\mathbf{e}_4$

- Weight $\mathbf{u}_\circ = \mathbb{G}\mathbf{u}$ All components with factor $\mathbf{e}_4$

Bulk and Weight of Point

$$\mathbf{p} = \underbrace{p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3}_{\text{Position}} + \underbrace{p_w\mathbf{e}_4}_{\text{Weight}}$$

$$\mathbf{p}_{\bullet} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$$

$$\mathbf{p}_{\circ} = p_w\mathbf{e}_4$$

Bulk and Weight of Line

$$\boldsymbol{l} = \underbrace{l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}}_{\text{Moment}}$$

$$\boldsymbol{l}_{\bullet} = l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$$

$$\boldsymbol{l}_{\circ} = l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}$$

Bulk and Weight of Plane

$$\mathbf{g} = \underbrace{g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w \mathbf{e}_{321}}_{\text{Position}}$$

$$\mathbf{g}_\bullet = g_w \mathbf{e}_{321}$$

$$\mathbf{g}_\circ = g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}$$

Bulk and Weight

- Bulk contains positional information
- Weight contains directional information
- If the bulk is zero, then the object contains the origin
- If the weight zero, then the horizon contains the object

Inner Product

- Dot product defined by metric: $\mathbf{a} \bullet \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}}\mathbf{G}\mathbf{b}\right)\mathbf{1}$

- Antidot product defined by antimetric: $\mathbf{a} \circ \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}}\mathbb{G}\mathbf{b}\right)\mathbb{1}$

- Satisfies De Morgan law: $\mathbf{a} \circ \mathbf{b} = \overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}}$

Bulk and Weight Norms

- Two dot products produce two norms
- Bulk norm: $\left\|\mathbf{u}\right\|_{\bullet} = \sqrt{\mathbf{u} \bullet \mathbf{u}}$
- Weight norm: $\left\|\mathbf{u}\right\|_{\circ} = \sqrt{\mathbf{u} \circ \mathbf{u}}$

Bulk and Weight Norms

Type	Bulk Norm	Weight Norm
Point **p**	$\lVert\mathbf{p}\rVert_\bullet = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2}$	$\lVert\mathbf{p}\rVert_\circ = \lvert p_w \rvert \mathbb{1}$
Line ***l***	$\lVert\boldsymbol{l}\rVert_\bullet = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$	$\lVert\boldsymbol{l}\rVert_\circ = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$
Plane **g**	$\lVert\mathbf{g}\rVert_\bullet = \lvert g_w \rvert \mathbf{1}$	$\lVert\mathbf{g}\rVert_\circ = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$

Unitization

- An object is *unitized* when its weight has magnitude one

Type	Definition	Unitization
Point $\mathbf{p}$	$\mathbf{p} = p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3 + p_w\,\mathbf{e}_4$	$p_w^2 = 1$
Line $\boldsymbol{l}$	$\boldsymbol{l} = l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43} + l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$	$l_{vx}^2 + l_{vy}^2 + l_{vz}^2 = 1$
Plane $\mathbf{g}$	$\mathbf{g} = g_x\,\mathbf{e}_{423} + g_y\,\mathbf{e}_{431} + g_z\,\mathbf{e}_{412} + g_w\,\mathbf{e}_{321}$	$g_x^2 + g_y^2 + g_z^2 = 1$

Geometric Norm

- Bulk and weight norms by themselves not meaningful
- But add them, and result is a *homogeneous magnitude*
- Represents distance from origin
- Called the geometric norm

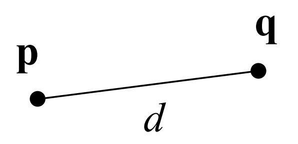
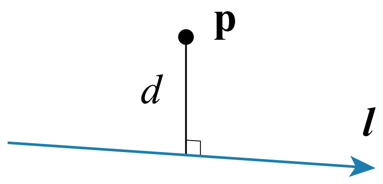
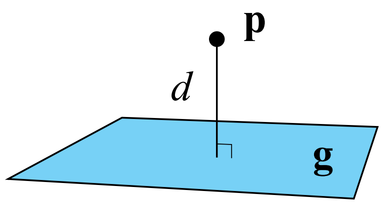
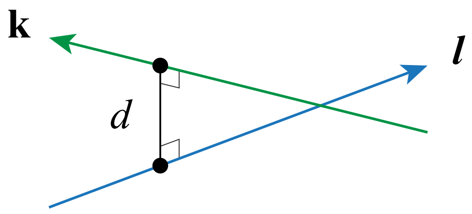
$$\|\mathbf{u}\| = \|\mathbf{u}\|_{\bullet} + \|\mathbf{u}\|_{\circ} = \sqrt{\mathbf{u} \bullet \mathbf{u}} + \sqrt{\mathbf{u} \circ \mathbf{u}}$$

- Can be unitized by making weight one

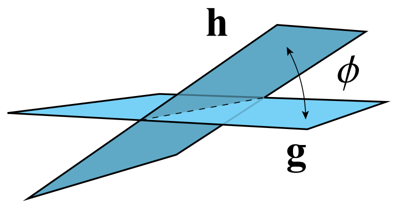
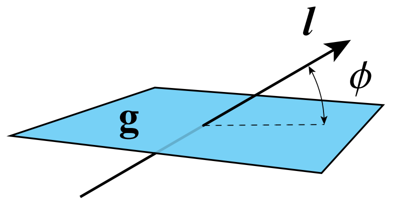
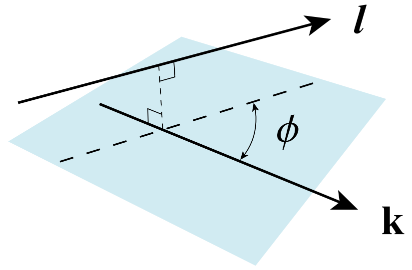
Geometric Norm

Type	Geometric Norm	Interpretation
Point **p**	$\left\lVert \widehat{\mathbf{p}} \right\rVert = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from the origin to the point **p**.
Line ***l***	$\left\lVert \widehat{\boldsymbol{l}} \right\rVert = \dfrac{\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}}{\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$	Perpendicular distance from the origin to the line ***l***.
Plane **g**	$\left\lVert \widehat{\mathbf{g}} \right\rVert = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$	Perpendicular distance from the origin to the plane **g**.

Euclidean Distance

Distance Formula	Illustration
Distance d between points $\mathbf{p}$ and $\mathbf{q}$. $d(\mathbf{p},\mathbf{q}) = \lVert \mathbf{q}_{xyz} p_w - \mathbf{p}_{xyz} q_w \rVert \mathbf{1} + \lvert p_w q_w \rvert \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and line $\boldsymbol{l}$. $d(\mathbf{p},\boldsymbol{l}) = \lVert \boldsymbol{l}_\mathbf{v} \times \mathbf{p}_{xyz} + p_w \boldsymbol{l}_\mathbf{m} \rVert \mathbf{1} + \lVert p_w \boldsymbol{l}_\mathbf{v} \rVert \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and plane $\mathbf{g}$. $d(\mathbf{p},\mathbf{g}) = (\mathbf{p}\cdot\mathbf{g})\mathbf{1} + \lVert p_w \mathbf{g}_{xyz} \rVert \mathbb{1}$	
Perpendicular distance d between skew lines $\boldsymbol{l}$ and $\mathbf{k}$. $d(\boldsymbol{l},\mathbf{k}) = -(\boldsymbol{l}_\mathbf{v}\cdot\mathbf{k}_\mathbf{m} + \boldsymbol{l}_\mathbf{m}\cdot\mathbf{k}_\mathbf{v})\mathbf{1} + \lVert \boldsymbol{l}_\mathbf{v} \times \mathbf{k}_\mathbf{v} \rVert \mathbb{1}$	

Euclidean Angle

Angle Formula	Illustration
Cosine of angle ϕ between planes $\mathbf{g}$ and $\mathbf{h}$. $\cos\phi(\mathbf{g},\mathbf{h}) = (\mathbf{g}_{xyz}\cdot\mathbf{h}_{xyz})\mathbf{1}+\|\mathbf{g}\|_{\circ}\|\mathbf{h}\|_{\circ}$	
Cosine of angle ϕ between plane $\mathbf{g}$ and line $\boldsymbol{l}$. $\cos\phi(\mathbf{g},\boldsymbol{l}) = \|\mathbf{g}_{xyz}\times\boldsymbol{l}_{\mathrm{v}}\|\mathbf{1}+\|\mathbf{g}\|_{\circ}\|\boldsymbol{l}\|_{\circ}$	
Cosine of angle ϕ between lines $\boldsymbol{l}$ and $\mathbf{k}$. $\cos\phi(\boldsymbol{l},\mathbf{k}) = (\boldsymbol{l}_{\mathrm{v}}\cdot\mathbf{k}_{\mathrm{v}})\mathbf{1}+\|\boldsymbol{l}\|_{\circ}\|\mathbf{k}\|_{\circ}$	

Bulk and Weight Duals

- Multiply by metric or antimetric, then take complement
- Bulk dual: $\mathbf{u}^{\bigstar} = \overline{\mathbf{G}\mathbf{u}}$
- Weight dual: $\mathbf{u}^{\star} = \overline{\mathbb{G}\mathbf{u}}$

Bulk and Weight Duals

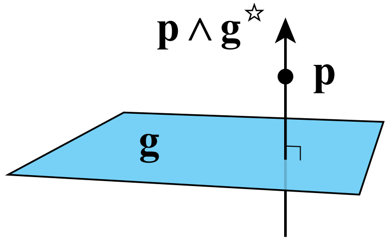
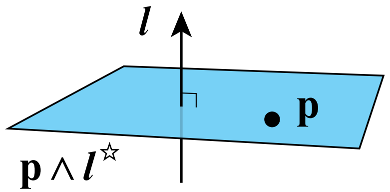
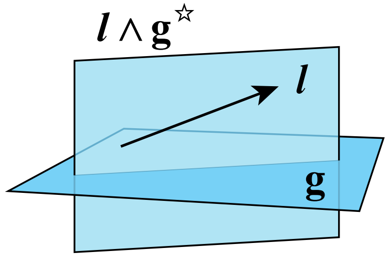
$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{u}^\bigstar$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0
$\mathbf{u}_\bigstar$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$\mathbf{e}_4$	0
$\mathbf{u}^\star$	0	0	0	0	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	0	$\mathbf{1}$
$\mathbf{u}_\star$	0	0	0	0	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	$\mathbf{1}$

Interior Products

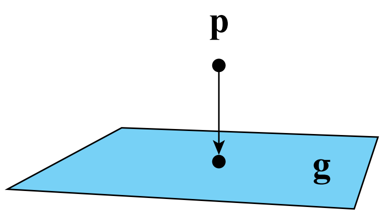
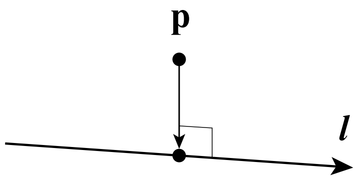
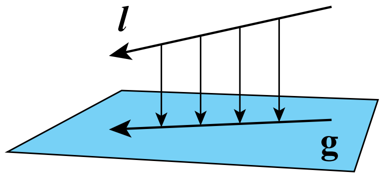
- Two exterior products combined with two duals
- Four *interior* products

- Bulk contraction $\mathbf{a} \vee \mathbf{b}^{\bigstar}$
- Weight contraction $\mathbf{a} \vee \mathbf{b}^{\,☆}$
- Bulk expansion $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$
- Weight expansion $\mathbf{a} \wedge \mathbf{b}^{\,☆}$

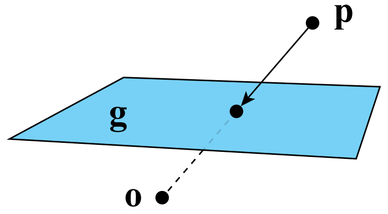
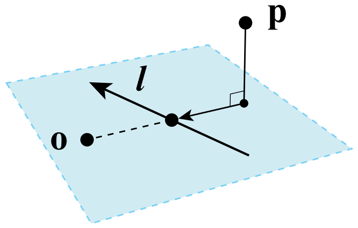
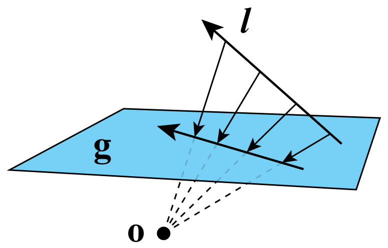
Weight Expansion

Expansion Operation	Illustration
Line containing point **p** and orthogonal to plane **g**. $$\mathbf{p} \wedge \mathbf{g}^{\star} = -p_w g_x \mathbf{e}_{41} - p_w g_y \mathbf{e}_{42} - p_w g_z \mathbf{e}_{43} + (p_z g_y - p_y g_z)\mathbf{e}_{23} + (p_x g_z - p_z g_x)\mathbf{e}_{31} + (p_y g_x - p_x g_y)\mathbf{e}_{12}$$	
Plane containing point **p** and orthogonal to line ***l***. $$\mathbf{p} \wedge \boldsymbol{l}^{\star} = -p_w l_{vx} \mathbf{e}_{423} - p_w l_{vy} \mathbf{e}_{431} - p_w l_{vz} \mathbf{e}_{412} + (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\mathbf{e}_{321}$$	
Plane containing line ***l*** and orthogonal to plane **g**. $$\boldsymbol{l} \wedge \mathbf{g}^{\star} = (l_{vy} g_z - l_{vz} g_y)\mathbf{e}_{423} + (l_{vz} g_x - l_{vx} g_z)\mathbf{e}_{431} + (l_{vx} g_y - l_{vy} g_x)\mathbf{e}_{412} - (l_{mx} g_x + l_{my} g_y + l_{mz} g_z)\mathbf{e}_{321}$$	

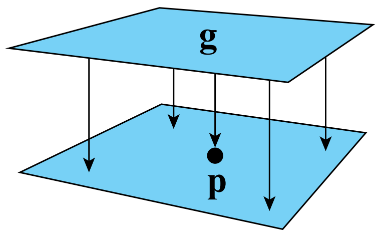
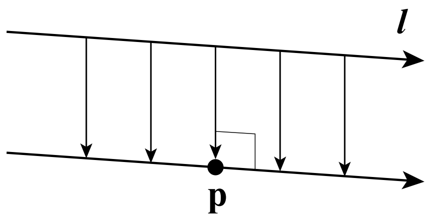
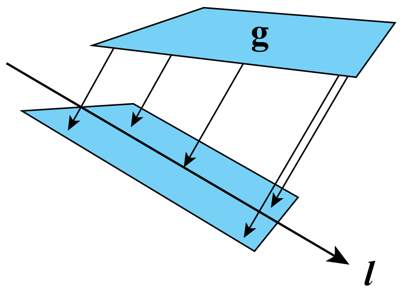
Orthogonal Projection

<table>
<tr><th>Projection Operation</th><th>Illustration</th></tr>
<tr><td>Orthogonal projection of point p onto plane g.
$$\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{\star}) = (g_x^2 + g_y^2 + g_z^2)(p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4) - (g_x p_x + g_y p_y + g_z p_z + g_w p_w)(g_x \mathbf{e}_1 + g_y \mathbf{e}_2 + g_z \mathbf{e}_3)$$</td><td></td></tr>
<tr><td>Orthogonal projection of point p onto line l.
$$\boldsymbol{l} \vee (\mathbf{p} \wedge \boldsymbol{l}^{\star}) = (l_{vx} p_x + l_{vy} p_y + l_{vz} p_z)(l_{vx} \mathbf{e}_1 + l_{vy} \mathbf{e}_2 + l_{vz} \mathbf{e}_3) + (l_{vx}^2 + l_{vy}^2 + l_{vz}^2)\, p_w \mathbf{e}_4 + (l_{vy} l_{mz} - l_{vz} l_{my})\, p_w \mathbf{e}_1 + (l_{vz} l_{mx} - l_{vx} l_{mz})\, p_w \mathbf{e}_2 + (l_{vx} l_{my} - l_{vy} l_{mx})\, p_w \mathbf{e}_3$$</td><td></td></tr>
<tr><td>Orthogonal projection of line l onto plane g.
$$\mathbf{g} \vee (\boldsymbol{l} \wedge \mathbf{g}^{\star}) = (g_x^2 + g_y^2 + g_z^2)(l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43}) - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})(g_x \mathbf{e}_{41} + g_y \mathbf{e}_{42} + g_z \mathbf{e}_{43}) + (g_x l_{mx} + g_y l_{my} + g_z l_{mz})(g_x \mathbf{e}_{23} + g_y \mathbf{e}_{31} + g_z \mathbf{e}_{12}) + (g_z l_{vy} - g_y l_{vz})\, g_w \mathbf{e}_{23} + (g_x l_{vz} - g_z l_{vx})\, g_w \mathbf{e}_{31} + (g_y l_{vx} - g_x l_{vy})\, g_w \mathbf{e}_{12}$$</td><td></td></tr>
</table>

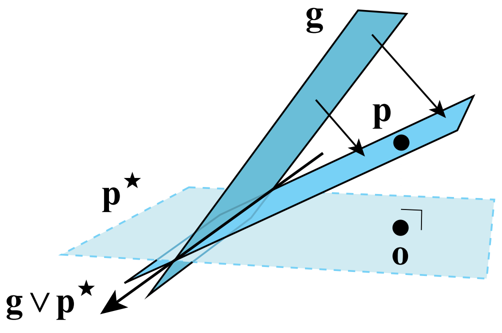
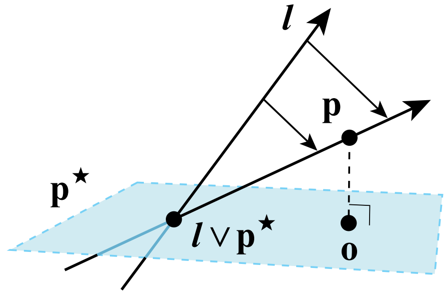
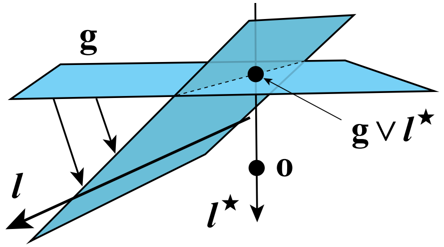
Central Projection

Projection Operation	Illustration
Central projection of point **p** onto plane **g**. $\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{\star}) = g_w^2 \left(p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 \right) - \left(g_x p_x + g_y p_y + g_z p_z \right) g_w \mathbf{e}_4$	
Central projection of point **p** onto line ***l***. $\boldsymbol{l} \vee (\mathbf{p} \wedge \boldsymbol{l}^{\star}) = \left(l_{mx}^2 + l_{my}^2 + l_{mz}^2 \right) \left(p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 \right) - \left(l_{mx} p_x + l_{my} p_y + l_{mz} p_z \right) \left(l_{mx} \mathbf{e}_1 + l_{my} \mathbf{e}_2 + l_{mz} \mathbf{e}_3 \right) + \left(l_{mx} \left(l_{vz} p_y - l_{vy} p_z \right) + l_{my} \left(l_{vx} p_z - l_{vz} p_x \right) + l_{mz} \left(l_{vy} p_x - l_{vx} p_y \right) \right) \mathbf{e}_4$	
Central projection of line ***l*** onto plane **g**. $\mathbf{g} \vee (\boldsymbol{l} \wedge \mathbf{g}^{\star}) = \left(g_y l_{mz} - g_z l_{my} \right) g_w \mathbf{e}_{41} + g_w^2 l_{mx} \mathbf{e}_{23} + \left(g_z l_{mx} - g_x l_{mz} \right) g_w \mathbf{e}_{42} + g_w^2 l_{my} \mathbf{e}_{31} + \left(g_x l_{my} - g_y l_{mx} \right) g_w \mathbf{e}_{43} + g_w^2 l_{mz} \mathbf{e}_{12}$	

Orthogonal Antiprojection

Projection Operation	Illustration
Orthogonal antiprojection of plane **g** onto point **p**. $\begin{aligned}\mathbf{p}\wedge\left(\mathbf{g}\vee\mathbf{p}^{\star}\right) &= p_w^2\left(g_x\mathbf{e}_{423}+g_y\mathbf{e}_{431}+g_z\mathbf{e}_{412}\right)\\ &\quad -\left(p_xg_x+p_yg_y+p_zg_z\right)p_w\mathbf{e}_{321}\end{aligned}$	
Orthogonal antiprojection of line ***l*** onto point **p**. $\begin{aligned}\mathbf{p}\wedge\left(\boldsymbol{l}\vee\mathbf{p}^{\star}\right) &= p_w^2\left(l_{vx}\mathbf{e}_{41}+l_{vy}\mathbf{e}_{42}+l_{vz}\mathbf{e}_{43}\right)\\ &\quad +\left(p_yl_{vz}-p_zl_{vy}\right)p_w\mathbf{e}_{23}\\ &\quad +\left(p_zl_{vx}-p_xl_{vz}\right)p_w\mathbf{e}_{31}\\ &\quad +\left(p_xl_{vy}-p_yl_{vx}\right)p_w\mathbf{e}_{12}\end{aligned}$	
Orthogonal antiprojection of plane **g** onto line ***l***. $\begin{aligned}\boldsymbol{l}\wedge\left(\mathbf{g}\vee\boldsymbol{l}^{\star}\right) &= \left(l_{vx}^2+l_{vy}^2+l_{vz}^2\right)\left(g_x\mathbf{e}_{423}+g_y\mathbf{e}_{431}+g_z\mathbf{e}_{412}\right)\\ &\quad -\left(l_{vx}g_x+l_{vy}g_y+l_{vz}g_z\right)\left(l_{vx}\mathbf{e}_{423}+l_{vy}\mathbf{e}_{431}+l_{vz}\mathbf{e}_{412}\right)\\ &\quad +\left(l_{vz}l_{my}-l_{vy}l_{mz}\right)g_x\mathbf{e}_{321}\\ &\quad +\left(l_{vx}l_{mz}-l_{vz}l_{mx}\right)g_y\mathbf{e}_{321}\\ &\quad +\left(l_{vy}l_{mx}-l_{vx}l_{my}\right)g_z\mathbf{e}_{321}\end{aligned}$	

Central Antiprojection

Projection Operation	Illustration
Central antiprojection of plane **g** onto point **p**. $\mathbf{p} \wedge \left(\mathbf{g} \vee \mathbf{p}^{\star}\right) = \left[\left(p_y^2 + p_z^2\right) g_x - \left(p_y g_y + p_z g_z + p_w g_w\right) p_x\right] \mathbf{e}_{423}$ $+ \left[\left(p_z^2 + p_x^2\right) g_y - \left(p_x g_x + p_z g_z + p_w g_w\right) p_y\right] \mathbf{e}_{431}$ $+ \left[\left(p_x^2 + p_y^2\right) g_z - \left(p_x g_x + p_y g_y + p_w g_w\right) p_z\right] \mathbf{e}_{412}$ $+ \left(p_x^2 + p_y^2 + p_z^2\right) g_w \mathbf{e}_{321}$	
Central antiprojection of line ***l*** onto point **p**. $\mathbf{p} \wedge \left(\boldsymbol{l} \vee \mathbf{p}^{\star}\right) = \left(p_x l_{vx} + p_y l_{vy} + p_z l_{vz}\right)\left(p_x \mathbf{e}_{41} + p_y \mathbf{e}_{42} + p_z \mathbf{e}_{43}\right)$ $+ \left(p_y^2 + p_z^2\right) l_{mx} \mathbf{e}_{23} + \left(p_z^2 + p_x^2\right) l_{my} \mathbf{e}_{31} + \left(p_x^2 + p_y^2\right) l_{mz} \mathbf{e}_{12}$ $+ \left(p_z l_{my} - p_y l_{mz}\right) p_w \mathbf{e}_{41} - \left(p_y l_{my} + p_z l_{mz}\right) p_x \mathbf{e}_{23}$ $+ \left(p_x l_{mz} - p_z l_{mx}\right) p_w \mathbf{e}_{42} - \left(p_z l_{mz} + p_x l_{mx}\right) p_y \mathbf{e}_{31}$ $+ \left(p_y l_{mx} - p_x l_{my}\right) p_w \mathbf{e}_{43} - \left(p_x l_{mx} + p_y l_{my}\right) p_z \mathbf{e}_{12}$	
Central antiprojection of plane **g** onto line ***l***. $\boldsymbol{l} \wedge \left(\mathbf{g} \vee \boldsymbol{l}^{\star}\right) = \left(l_{mx} g_x + l_{my} g_y + l_{mz} g_z\right)\left(l_{mx} \mathbf{e}_{423} + l_{my} \mathbf{e}_{431} + l_{mz} \mathbf{e}_{412}\right)$ $+ \left(l_{my} l_{vz} - l_{mz} l_{vy}\right) g_w \mathbf{e}_{423} + \left(l_{mz} l_{vx} - l_{mx} l_{vz}\right) g_w \mathbf{e}_{431}$ $+ \left(l_{mx} l_{vy} - l_{my} l_{vx}\right) g_w \mathbf{e}_{412} + \left(l_{mx}^2 + l_{my}^2 + l_{mz}^2\right) g_w \mathbf{e}_{321}$	

Geometric Product

- Historically denoted by juxtaposition without symbol
- But there is always product and antiproduct
- We use upward and downward wedge with dot inside

- Geometric product $\mathbf{a} \mathbin{\dot\wedge} \mathbf{b}$

- Geometric antiproduct $\mathbf{a} \mathbin{\dot\vee} \mathbf{b}$

- “Wedge-dot” and “Antiwedge-dot”

Geometric Product

- Defined by slightly different property compared to exterior product
- For vectors, $\mathbf{v} \mathbin{\wedge\!\!\!\!\cdot\,} \mathbf{v} = \mathbf{v} \bullet \mathbf{v}$
- Geometric product depends on the metric

4D Geometric Product

Geometric Product $\mathbf{a} ⟑ \mathbf{b}$

a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	$\mathbf{e}_3$	$-\mathbf{e}_{321}$	$-\mathbf{e}_1$	$-\mathbf{e}_{43}$	$\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{1}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbb{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_4$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	$-\mathbf{e}_{423}$	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$\mathbf{e}_4$	$\mathbf{e}_{423}$	0	0	0	0	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	$-\mathbf{e}_{431}$	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbf{e}_4$	0	0	0	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	$-\mathbf{e}_{412}$	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$-\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$-\mathbf{e}_{321}$	$-\mathbf{e}_1$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	$-\mathbf{1}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	0	0	0	$\mathbf{e}_{41}$	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	0	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	0	0	0	$\mathbf{e}_{42}$	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	0	0	0	$\mathbf{e}_{43}$	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{1}$	$\mathbf{e}_4$
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0

Geometric Antiproduct

- Defined by De Morgan law:

$$\mathbf{a} \mathbin{\dot{\vee}} \mathbf{b} = \overline{\underline{\mathbf{a}} \mathbin{\dot{\wedge}} \underline{\mathbf{b}}}$$

- Antivector **u** squares to antidot product:

$$\mathbf{u} \mathbin{\dot{\vee}} \mathbf{u} = \mathbf{u} \circ \mathbf{u}$$

4D Geometric Antiproduct

Geometric Antiproduct $\mathbf{a} ⟇ \mathbf{b}$

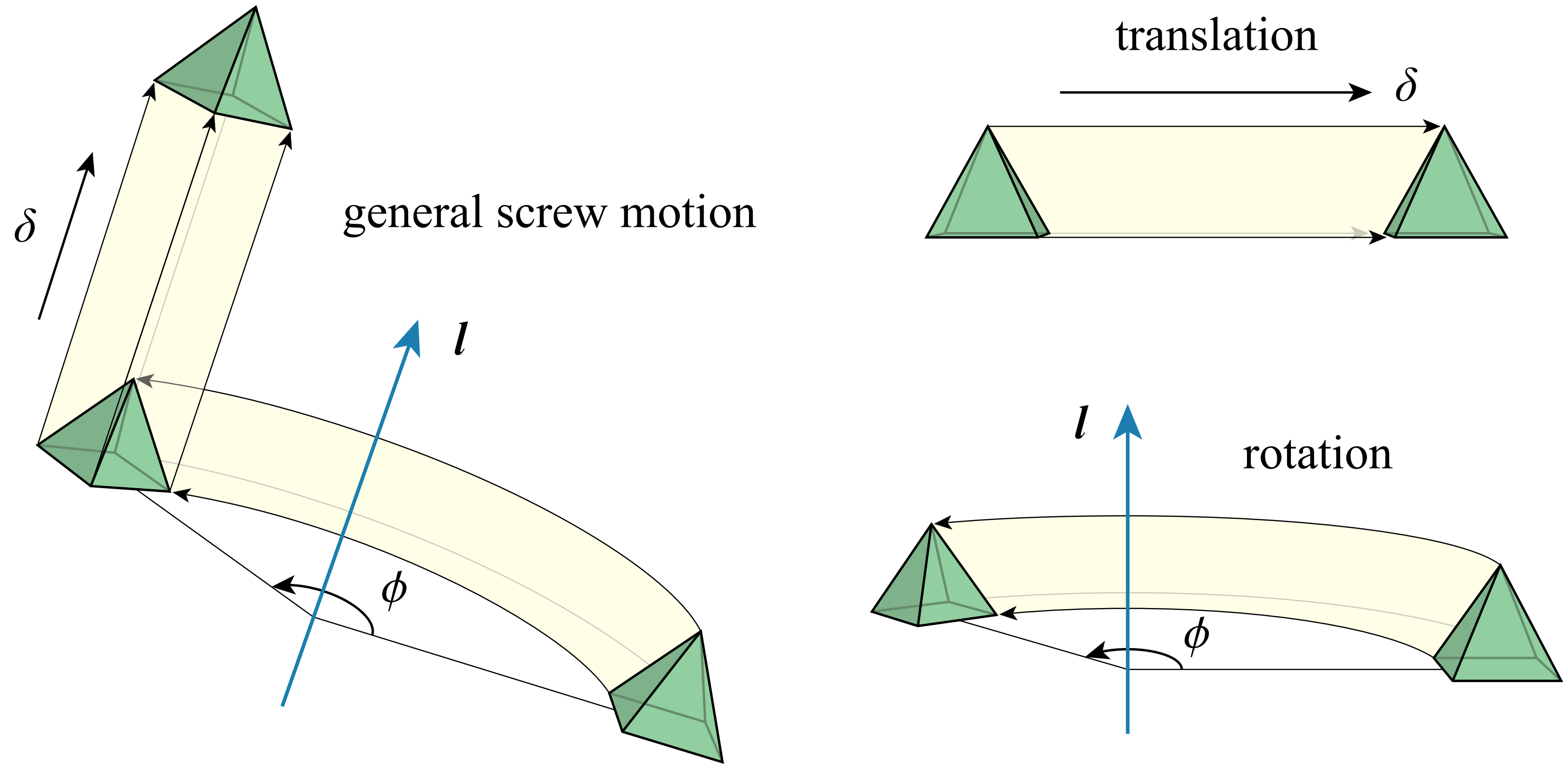
$\mathbf{a}$ \ $\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	0	$\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	0	0	$\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$	$-\mathbf{e}_{321}$	$\mathbf{e}_1$	0	0	0	$\mathbf{e}_{12}$	$\mathbf{1}$	$-\mathbf{e}_{23}$	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	$-\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{321}$	0	0	0	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{1}$	0	$\mathbf{e}_3$
$\mathbf{e}_4$	$-\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{1}$	$\mathbf{e}_4$
$\mathbf{e}_{41}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	$-\mathbf{e}_{321}$	$\mathbf{e}_1$	$\mathbf{e}_{431}$	$-\mathbf{e}_{43}$	$-\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_4$	$\mathbf{e}_{423}$	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbb{1}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	$\mathbf{e}_1$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	0	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	0	0	0	$-\mathbf{e}_3$	$-\mathbf{e}_{321}$	$\mathbf{e}_1$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	$-\mathbf{e}_1$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	$-\mathbf{e}_2$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_4$	$\mathbf{e}_{423}$	$\mathbf{e}_3$	$\mathbf{e}_{321}$	$-\mathbf{e}_1$	$\mathbf{e}_{43}$	$\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	$-\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{321}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbb{1}$	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$-\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Geometric Product

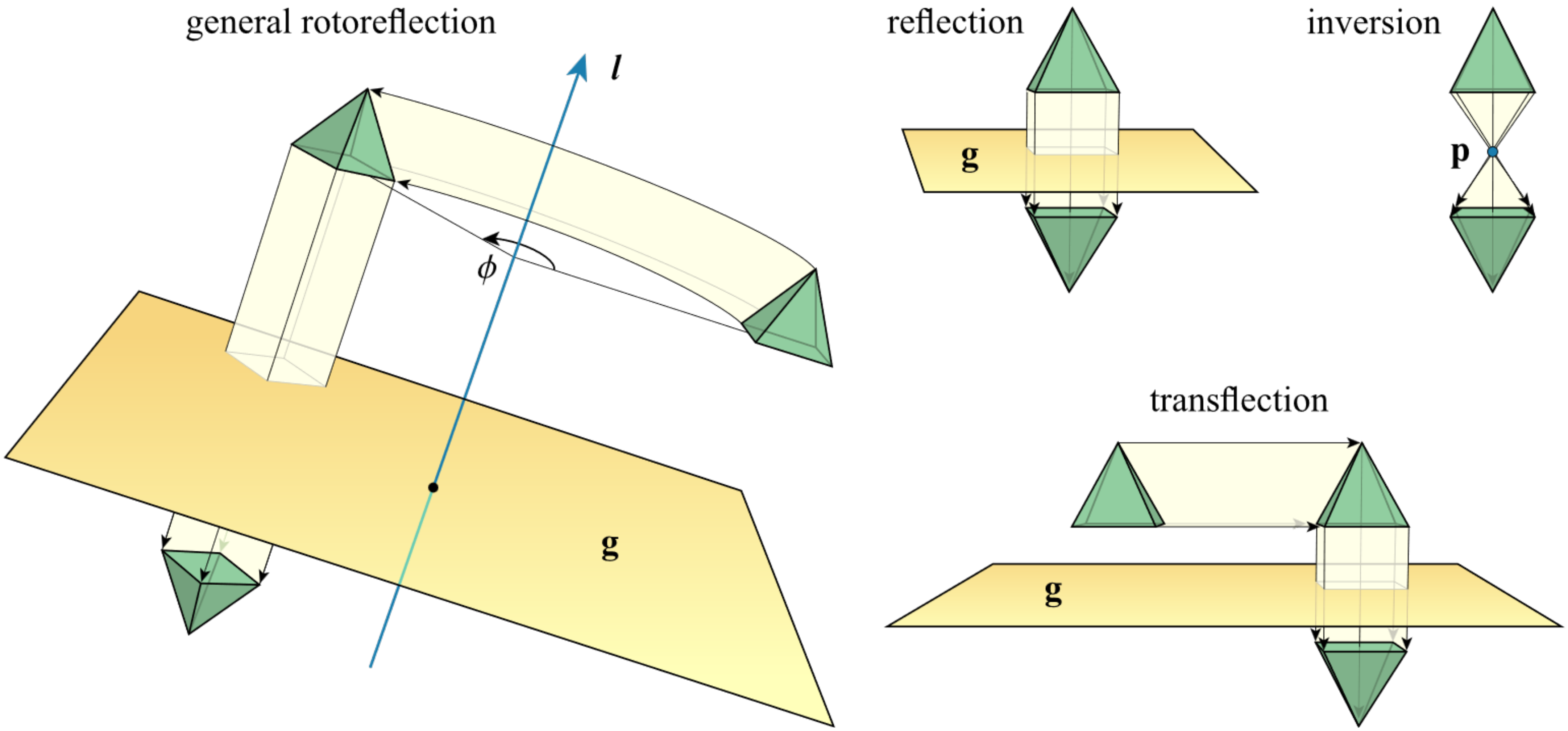
- Geometric product in 4D space fixes the origin
- Cannot perform transformations we want

- Geometric antiproduct performs Euclidean isometries
- Uses sandwiching similar to quaternions

Proper Euclidean Isometries



Improper Euclidean Isometries



Reverse and Antireverse

- Reverse $\tilde{\mathbf{u}}$ multiplies vectors in reverse order
  - (with geometric product)

- Antireverse $\underset{\sim}{\mathbf{u}}$ multiplies antivectors in reverse order
  - (with geometric antiproduct)

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\tilde{\mathbf{u}}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$\mathbb{1}$
$\underset{\sim}{\mathbf{u}}$	$\mathbf{1}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Geometric Antiproduct

- Sandwiches with geometric antiproduct perform Euclidean isometries

- Motor = MOtion operaTOR

- Flector = reFLECtion operaTOR

Motor

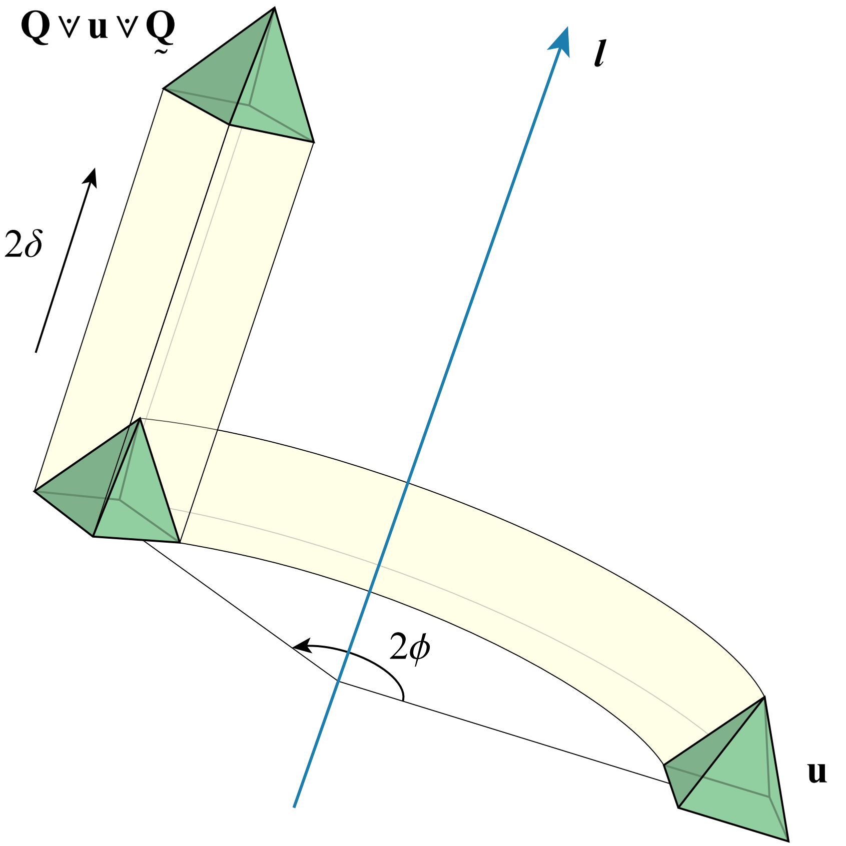
- General form of a motor:

$$\mathbf{Q} = \underbrace{Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1}}_{\text{Rotation Quaternion}} + \underbrace{Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}}_{\text{Moment and Displacement}}$$

- Performs any combination of rotations and translations

$$\mathbf{u}' = \mathbf{Q} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{Q}}$$

Motor



$$\mathbf{Q} = \exp_{\dot\vee}\left[\left(\delta\mathbf{1} + \varphi\mathbb{1}\right)\dot\vee\,\boldsymbol{l}\right] = \boldsymbol{l}\sin\varphi - \boldsymbol{l}^{\star}\delta\cos\varphi - \delta\sin\varphi + \mathbb{1}\cos\varphi$$

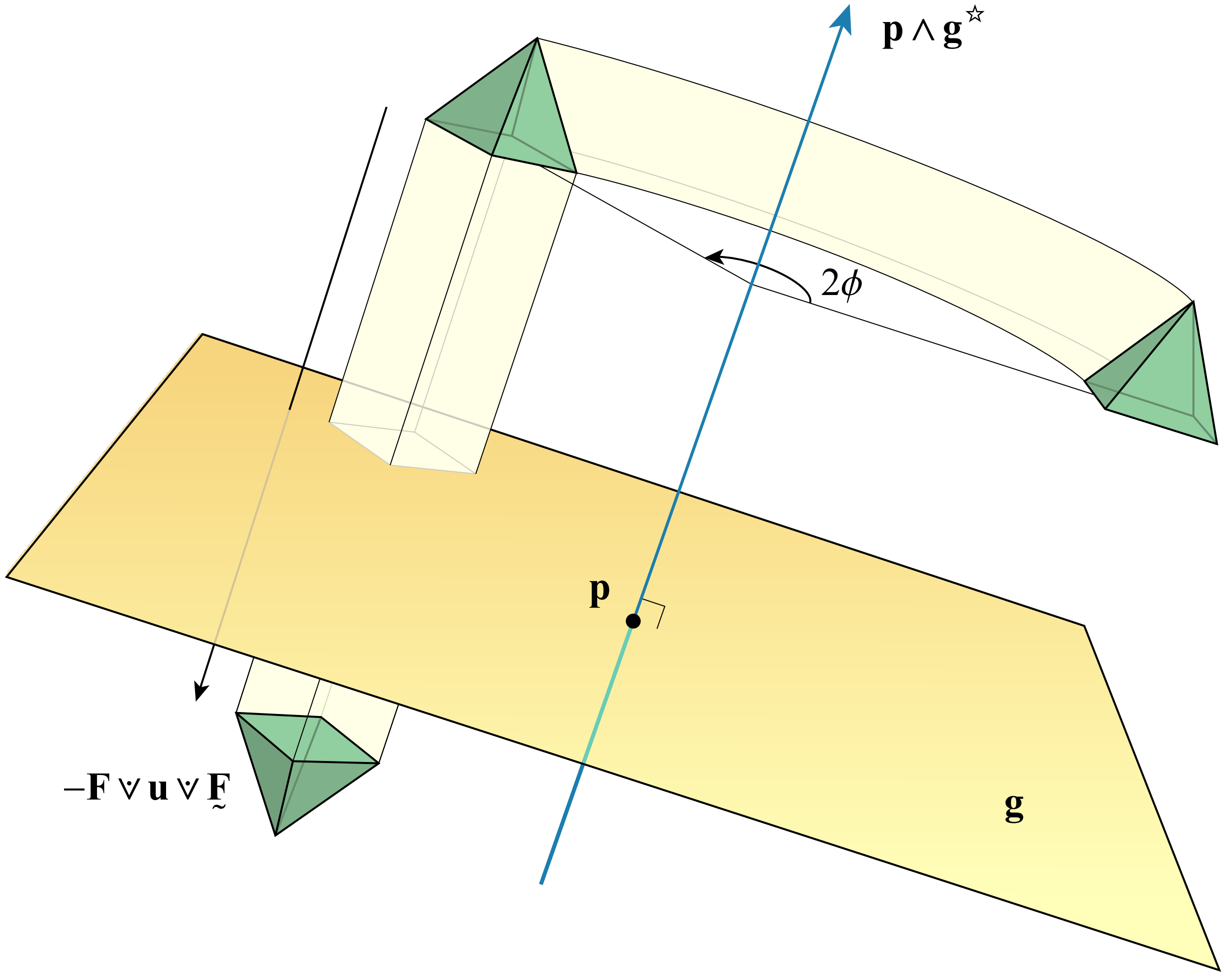
Flector

- General form of a flector:

$$\mathbf{F} = \underbrace{F_{px}\,\mathbf{e}_1 + F_{py}\,\mathbf{e}_2 + F_{pz}\,\mathbf{e}_3 + F_{pw}\,\mathbf{e}_4}_{\text{Point}} + \underbrace{F_{gx}\,\mathbf{e}_{423} + F_{gy}\,\mathbf{e}_{431} + F_{gz}\,\mathbf{e}_{412} + F_{gw}\,\mathbf{e}_{321}}_{\text{Plane}}$$

- Performs any combination of rotoreflections

Flector



$$\mathbf{F} = \mathbf{p}\sin\varphi + \mathbf{g}\cos\varphi$$

Motor-Point Transformation

- 25 multiply-adds:

$$\mathbf{p}'_{xyz} = \mathbf{p}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a} - Q_{mw}\,p_w\,\mathbf{v}\right)$$
$$p'_w = p_w$$

$$\mathbf{a} = \mathbf{v} \times \mathbf{p}_{xyz} + p_w\mathbf{m}$$

$$\mathbf{v} = \left(Q_{vx}, Q_{vy}, Q_{vz}\right)$$
$$\mathbf{m} = \left(Q_{mx}, Q_{my}, Q_{mz}\right)$$

- 3x4 matrix transformation only requires 12 multiply-adds

Motor-Line Transformation

- 54 multiply-adds:

$$\boldsymbol{l}'_{\mathbf{v}} = \boldsymbol{l}_{\mathbf{v}} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a}\right)$$

$$\boldsymbol{l}'_{\mathbf{m}} = \boldsymbol{l}_{\mathbf{m}} + 2\left[Q_{mw}\mathbf{a} + Q_{vw}\left(\mathbf{b} + \mathbf{c}\right) + \mathbf{v} \times \left(\mathbf{b} + \mathbf{c}\right) + \mathbf{m} \times \mathbf{a}\right]$$

$$\mathbf{a} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{v}} \qquad \mathbf{b} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{m}} \qquad \mathbf{c} = \mathbf{m} \times \boldsymbol{l}_{\mathbf{v}}$$

- 6x6 matrix transformation only requires 27 multiply-adds

Motor-Plane Transformation

- 35 multiply-adds:

$$\mathbf{g}'_{xyz} = \mathbf{g}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a}\right)$$

$$g'_w = g_w + 2\left[\left(\mathbf{m} \times \mathbf{g}_{xyz} + Q_{mw}\mathbf{g}_{xyz}\right) \cdot \mathbf{v} - Q_{vw}\left(\mathbf{m} \cdot \mathbf{g}_{xyz}\right)\right]$$

$$\mathbf{a} = \mathbf{v} \times \mathbf{g}_{xyz}$$

- 4x4 matrix transformation only requires 13 multiply-adds

Motor to Matrix

$$\mathbf{A_Q} = \begin{bmatrix} 1-2\left(Q_{vy}^2 + Q_{vz}^2\right) & 2Q_{vx}Q_{vy} & 2Q_{vz}Q_{vx} & 2\left(Q_{vy}Q_{mz} - Q_{vz}Q_{my}\right) \\ 2Q_{vx}Q_{vy} & 1-2\left(Q_{vz}^2 + Q_{vx}^2\right) & 2Q_{vy}Q_{vz} & 2\left(Q_{vz}Q_{mx} - Q_{vx}Q_{mz}\right) \\ 2Q_{vz}Q_{vx} & 2Q_{vy}Q_{vz} & 1-2\left(Q_{vx}^2 + Q_{vy}^2\right) & 2\left(Q_{vx}Q_{my} - Q_{vy}Q_{mx}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B_Q} = \begin{bmatrix} 0 & -2Q_{vz}Q_{vw} & 2Q_{vy}Q_{vw} & 2\left(Q_{vw}Q_{mx} - Q_{vx}Q_{mw}\right) \\ 2Q_{vz}Q_{vw} & 0 & -2Q_{vx}Q_{vw} & 2\left(Q_{vw}Q_{my} - Q_{vy}Q_{mw}\right) \\ -2Q_{vy}Q_{vw} & 2Q_{vx}Q_{vw} & 0 & 2\left(Q_{vw}Q_{mz} - Q_{vz}Q_{mw}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M_Q} = \mathbf{A_Q} + \mathbf{B_Q} \qquad \mathbf{M_Q^{-1}} = \mathbf{A_Q} - \mathbf{B_Q}$$

Motor Composition

- 48 multiply-adds:

$$
\begin{aligned}
\mathbf{Q} \vee \mathbf{R} = &\left(Q_{vw}R_{vx} + Q_{vx}R_{vw} + Q_{vy}R_{vz} - Q_{vz}R_{vy}\right)\mathbf{e}_{41} \\
&+ \left(Q_{vw}R_{vy} - Q_{vx}R_{vz} + Q_{vy}R_{vw} + Q_{vz}R_{vx}\right)\mathbf{e}_{42} \\
&+ \left(Q_{vw}R_{vz} + Q_{vx}R_{vy} - Q_{vy}R_{vx} + Q_{vz}R_{vw}\right)\mathbf{e}_{43} \\
&+ \left(Q_{vw}R_{vw} - Q_{vx}R_{vx} - Q_{vy}R_{vy} - Q_{vz}R_{vz}\right)\mathbb{1} \\
&+ \left(Q_{mw}R_{vx} + Q_{mx}R_{vw} + Q_{my}R_{vz} - Q_{mz}R_{vy} + Q_{vw}R_{mx} + Q_{vx}R_{mw} + Q_{vy}R_{mz} - Q_{vz}R_{my}\right)\mathbf{e}_{23} \\
&+ \left(Q_{mw}R_{vy} - Q_{mx}R_{vz} + Q_{my}R_{vw} + Q_{mz}R_{vx} + Q_{vw}R_{my} - Q_{vx}R_{mz} + Q_{vy}R_{mw} + Q_{vz}R_{mx}\right)\mathbf{e}_{31} \\
&+ \left(Q_{mw}R_{vz} + Q_{mx}R_{vy} - Q_{my}R_{vx} + Q_{mz}R_{vw} + Q_{vw}R_{mz} + Q_{vx}R_{my} - Q_{vy}R_{mx} + Q_{vz}R_{mw}\right)\mathbf{e}_{12} \\
&+ \left(Q_{mw}R_{vw} - Q_{mx}R_{vx} - Q_{my}R_{vy} - Q_{mz}R_{vz} + Q_{vw}R_{mw} - Q_{vx}R_{mx} - Q_{vy}R_{my} - Q_{vz}R_{mz}\right)\mathbf{1}
\end{aligned}
$$

- Composition of equiv 3x4 matrices requires 33 multiply-adds

Matrix Advantages

- Can represent more transformations
- Can read off origin and axis directions in transformed space
- Faster to transform objects
- Faster to compose

Motor Advantages

- Smaller storage requirements
  - Usually 8 floats, but can reduce to 6
- Inversion is trivial
  - Just reverse, negating bivector components
- Better parameterization
- Better interpolation properties

Conformal Geometric Algebra

- Adds two extra projective dimensions
  - One is stereographic projection
- Can represent round objects
  - With real, imaginary, or null radii
- Flat objects are round objects with infinite radius

Round Point

- Vector

$$\mathbf{a} = \underbrace{a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3 + a_w \mathbf{e}_4}_{\text{Carrier Point}} + \underbrace{a_u \mathbf{e}_5}_{\text{Infinity}},$$

$\left(\text{when } a_x = a_y = a_z = a_w = 0\right)$

Dipole

- Bivector

$$\mathbf{d} = \underbrace{\overbrace{d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43}}^{\text{Cocarrier Normal}} + d_{mx}\mathbf{e}_{23} + d_{my}\mathbf{e}_{31} + d_{mz}\mathbf{e}_{12}}_{\text{Carrier Line}} + \underbrace{d_{px}\mathbf{e}_{15} + d_{py}\mathbf{e}_{25} + d_{pz}\mathbf{e}_{35} + \overbrace{d_{pw}\mathbf{e}_{45}}^{\text{Cocarrier Position}}}_{\text{Flat Point}}.$$

$\left(\text{when } d_{vz} = d_{vy} = d_{vz} = d_{mx} = d_{my} = d_{mz} = 0\right)$

Circle

- Trivector

$$\mathbf{c} = \overbrace{\underbrace{c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412} + c_{gw}\mathbf{e}_{321}}_{\text{Carrier Plane}}}^{\text{Cocarrier Direction}} + \overbrace{\underbrace{c_{vx}\mathbf{e}_{415} + c_{vy}\mathbf{e}_{425} + c_{vz}\mathbf{e}_{435} + c_{mx}\mathbf{e}_{235} + c_{my}\mathbf{e}_{315} + c_{mz}\mathbf{e}_{125}}_{\text{Flat Line}\ (\text{when } c_{gx} = c_{gy} = c_{gz} = c_{gw} = 0)}}^{\text{Cocarrier Moment}} .$$

Sphere

- Quadrivector

$$\mathbf{s} = \underbrace{s_u \mathbf{e}_{1234}}_{\text{Carrier Space}} + \underbrace{s_x \mathbf{e}_{4235} + s_y \mathbf{e}_{4315} + s_z \mathbf{e}_{4125} + s_w \mathbf{e}_{3215}}_{\substack{\text{Flat Plane} \\ (\text{when } s_u = 0)}},$$

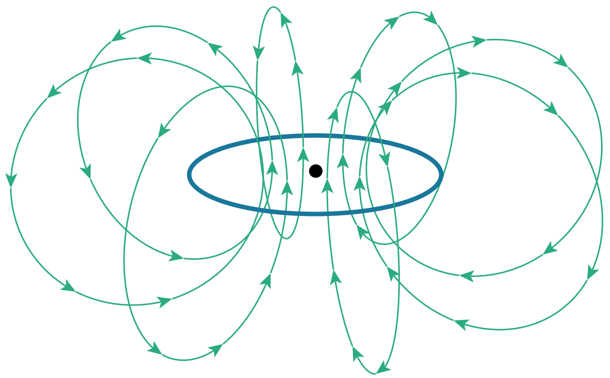
Operations

- Contains all operations from projective algebra
- Plus many conformal operations as well

Circle Rotations

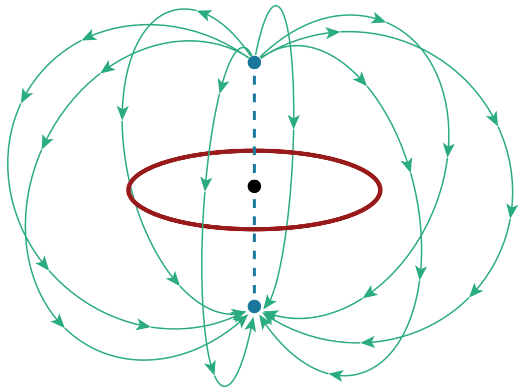
Real Circle / Elliptic Rotation

$$\mathbf{R} = \mathbf{c}\sin\phi + \mathbb{1}\cos\phi$$



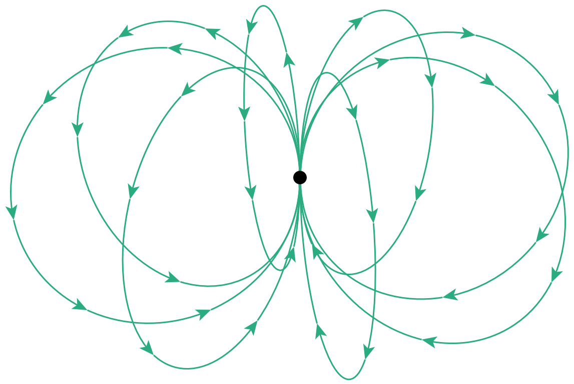
Imaginary Circle / Hyperbolic Rotation

$$\mathbf{R} = \mathbf{c}\sinh\phi + \mathbb{1}\cosh\phi$$



Null Circle / Parabolic Rotation

$$\mathbf{R} = \phi\mathbf{c} + \mathbb{1}$$



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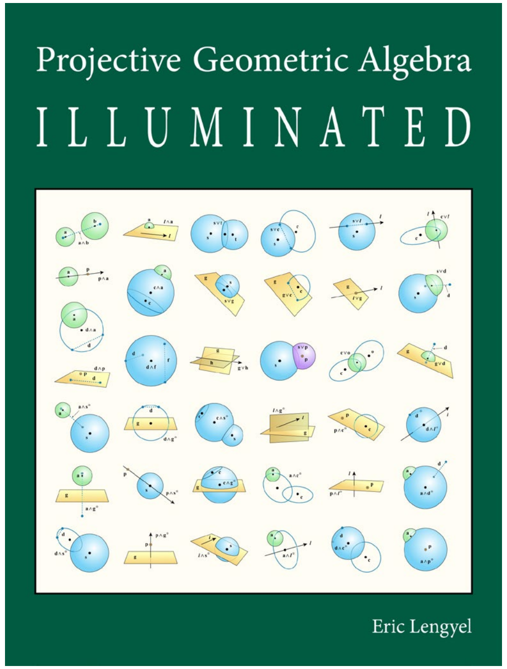

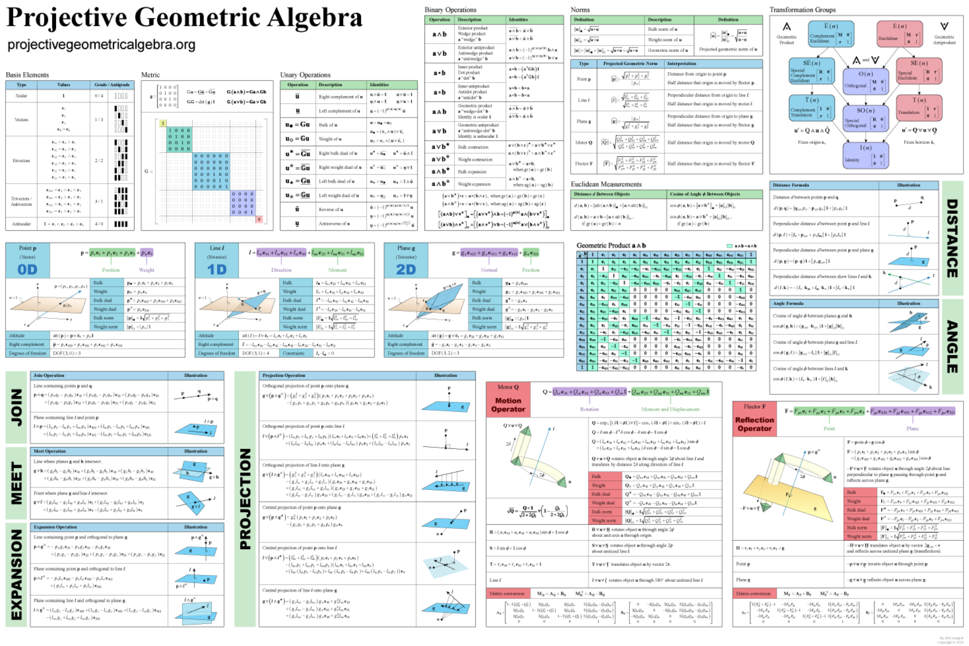

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