Hyperreal Structures Arising From An Infinite Base Logarithm

by

Eric Lengyel

Abstract

This paper presents new concepts in the use of infinite and infinitesimal numbers in real analysis. The theory is based upon the hyperreal number system developed by Abraham Robinson in the 1960's in his invention of "nonstandard analysis". The paper begins with a short exposition of the construction of the hyperreal number system and the fundamental results of nonstandard analysis which are used throughout the paper. The new theory which is built upon this foundation organizes the set of hyperreal numbers through structures which depend on an infinite base logarithm. Several new relations are introduced whose properties enable the simplification of calculations involving infinite and infinitesimal numbers. The paper explores two areas of application of these results to standard problems in elementary calculus. The first is to the evaluation of limits which assume certain indeterminate forms. The second is to the determination of convergence of infinite series. Both applications provide methods which greatly reduce the amount of computation necessary in many situations.

Overview

In the 1960's, Abraham Robinson developed what is called “nonstandard analysis”, and in doing so provided a rigorous foundation for the use of infinitesimals in analysis. A new number system known as the set of hyperreal numbers was constructed which includes the set of real numbers and also contains infinite and infinitesimal numbers. This paper begins with the construction of the hyperreals as a set of equivalence classes of sequences of real numbers. Included in this introductory section is the method by which relations defined on the real numbers are extended to relations on the hyperreal numbers. Through these extensions, it is possible to prove statements that hold true over the hyperreal numbers which are the “nonstandard” equivalents of statements that hold true over the real numbers. In nonstandard analysis, the proofs of these statements are usually facilitated by the utilization of what is called the transfer principle. This concept, however, requires a great deal of development of rigorous logic which will not be needed in the remainder of the paper. Therefore, the transfer principle is not used in the introductory section at all, and alternate proofs of nonstandard results are instead given.

In Section 2, we begin the study of new hyperreal structures which organize the set of hyperreal numbers into classes called zones, and we introduce the notions of superiority and local equality. These concepts depend on the logarithms of the hyperreal numbers taken to a fixed infinite base. We will be using infinite and infinitesimal numbers in such a way that our calculations will not require knowledge of how the hyperreal number system was constructed. Nor will our calculations require us to know what specific member of the set of hyperreals that we are using as the infinite base of our logarithm—knowing only that the number is infinite will suffice.

Once the theory has been developed in Section 2, we proceed in Sections 3 and 4 to present applications in two areas. The first application is to the evaluation of certain limits which assume the indeterminate forms $0/0$, ∞/∞, 1^{∞}, and $\infty - \infty$. The methods presented will provide alternatives to l'Hôpital's rule which generally allow much more efficient computation. The second application is to the determination of the convergence of infinite series. Two new convergence tests will be presented which are analogous to the comparison test and limit comparison test. As with limit evaluation, these tests significantly reduce the amount of computation in many situations.

1 Introduction to Nonstandard Analysis

This section presents the reader with the fundamentals of nonstandard analysis which are used throughout the remainder of this paper. We begin with the ultrapower construction of the set of hyperreal numbers and then proceed to introduce several relations and algebraic structures which are defined on this set. This section consists only of the necessary background information which can be found in any introductory text on nonstandard analysis, most notably [3]. Many of the proofs included in this section can also be found in the literature, the exceptions being those which rely on the transfer principle.

1.1 Construction of the Hyperreal Number System

The goals of the construction of the hyperreal number system are to build a field which contains an isomorphic copy of the real numbers as a proper subfield and also contains infinite and infinitesimal numbers. Furthermore, it is desired that the numbers in this new field obey all of the same laws which hold true over the real numbers. A field having these properties is constructed by using a free ultrafilter to partition the set of all sequences of real numbers into equivalence classes. It is then these equivalence classes which are the elements of the set of hyperreal numbers.

Before presenting the actual construction of the hyperreals, we include the definition of a filter.

Definition 1.1. A *filter* $\mathcal{F}$ on a set S is a nonempty collection of subsets of S having the following properties.

(a) $\varnothing \notin \mathcal{F}$

(b) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$

(c) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq S$ then $B \in \mathcal{F}$

Note that by (c), a filter $\mathcal{F}$ on S always contains S, and that by (a) and (b), no two elements of $\mathcal{F}$ are disjoint.

An *ultrafilter* on an infinite set S is a maximal filter on S. The existence of an ultrafilter follows from Zorn's Lemma. A filter $\mathcal{F}$ on S is an ultrafilter if and only if it has the following property.

(d) If $A \subseteq S$ then either $A \in \mathcal{F}$ or $S - A \in \mathcal{F}$

An ultrafilter $\mathcal{U}$ on an infinite set S is called *fixed* or *principal* if there exists $a \in S$ such that $\mathcal{U} = \{A \subseteq S \mid a \in A\}$. Ultrafilters which are not fixed are called *free*. An important fact is that free ultrafilters cannot contain any finite sets. By (d), this implies that if $\mathcal{U}$ is free then every cofinite subset of S is contained in $\mathcal{U}$.

We now construct the set of hyperreal numbers and prove that it is a linearly ordered field. We begin the construction by choosing a free ultrafilter $\mathcal{U}$ on the set of natural numbers $\mathbb{N}$. The ultrafilter $\mathcal{U}$ is not explicitly defined since it does not matter which free ultrafilter on $\mathbb{N}$ that we use. The set of all free ultrafilters on $\mathbb{N}$ determines a set of isomorphic fields from which we can choose any member to be the set of hyperreal numbers. Using $\mathcal{U}$, the hyperreals are constructed by considering the set of all sequences of real numbers indexed by $\mathbb{N}$ and defining the following relation on this set.

Definition 1.2. Given two sequences of real numbers $\langle a_n \rangle$ and $\langle b_n \rangle$, $\langle a_n \rangle \equiv \langle b_n \rangle$ if and only if $\{n \in \mathbb{N} \mid a_n = b_n\} \in \mathcal{U}$. The entries of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are then said to be equal "almost everywhere".

The phrase "almost everywhere" (abbreviated a.e.) is used to signify that the entries of a sequence have a certain property on some set in the ultrafilter $\mathcal{U}$. For instance, if the sequence $\langle a_n \rangle$ has the property that a_n is an integer for all n in some member of $\mathcal{U}$, then a_n is said to be an integer almost everywhere.

Proposition 1.3. The relation $\equiv$ is an equivalence relation.

Proof. Let $\langle a_n \rangle$, $\langle b_n \rangle$, and $\langle c_n \rangle$ be sequences of real numbers. $\langle a_n \rangle \equiv \langle a_n \rangle$, and thus $\equiv$ is reflexive, since $\mathbb{N} \in \mathcal{U}$. Because equality on the reals is symmetric, if $\langle a_n \rangle \equiv \langle b_n \rangle$ then $\langle b_n \rangle \equiv \langle a_n \rangle$, and thus $\equiv$ is symmetric. For transitivity, suppose $\langle a_n \rangle \equiv \langle b_n \rangle$ and $\langle b_n \rangle \equiv \langle c_n \rangle$. Let $A = \{n \in \mathbb{N} \mid a_n = b_n\}$ and $B = \{n \in \mathbb{N} \mid b_n = c_n\}$. Then $A \cap B = \{n \in \mathbb{N} \mid a_n = c_n\}$. Since A and B are elements of the ultrafilter $\mathcal{U}$, $A \cap B \in \mathcal{U}$. Therefore, $\langle a_n \rangle \equiv \langle c_n \rangle$.

The set of hyperreal numbers, which is denoted by $^*\mathbb{R}$, is defined to be the set of all equivalence classes induced by the equivalence relation $\equiv$. We will use the notation $[\langle a_n \rangle]$ to represent the equivalence class containing $\langle a_n \rangle$.

Addition, multiplication, and an ordering are defined on $^*\mathbb{R}$ as follows.

Definition 1.4. Let $[\langle a_n\rangle],[\langle b_n\rangle]\in {}^*\mathbb{R}$. The operations + (addition) and · (multiplication) and the relation < (less than) are defined by

(a) $[\langle a_n\rangle]+[\langle b_n\rangle]=[\langle a_n+b_n\rangle]$

(b) $[\langle a_n\rangle]\cdot[\langle b_n\rangle]=[\langle a_n\cdot b_n\rangle]$

(c) $[\langle a_n\rangle]<[\langle b_n\rangle]$ if and only if $\{n\in\mathbb{N}\mid a_n<b_n\}\in\mathcal{U}$

Proposition 1.5. The operations + and ·, and the relation < are well-defined.

Proof. Suppose $[\langle a_n\rangle]=[\langle b_n\rangle]$ and $[\langle c_n\rangle]=[\langle d_n\rangle]$, and let $A=\{n\in\mathbb{N}\mid a_n=b_n\}$ and $C=\{n\in\mathbb{N}\mid c_n=d_n\}$ (thus $A,C\in\mathcal{U}$). Then $\{n\in\mathbb{N}\mid a_n+c_n=b_n+d_n\}=A\cap C\in\mathcal{U}$. So the two sums are equal elements of the hyperreals. The same argument holds true for multiplication.

Now suppose $[\langle a_n\rangle]<[\langle c_n\rangle]$ and let $K=\{n\in\mathbb{N}\mid a_n<c_n\}\in\mathcal{U}$. Then the set on which $b_n<c_n$ contains $K\cap A$ and is therefore in $\mathcal{U}$. So $[\langle b_n\rangle]<[\langle c_n\rangle]$. Let $L=\{n\in\mathbb{N}\mid b_n<c_n\}$. Then the set on which $b_n<d_n$ contains $L\cap C$ and is therefore in $\mathcal{U}$. Thus $[\langle b_n\rangle]<[\langle d_n\rangle]$.

We now claim that the set of hyperreal numbers combined with the operations and ordering given in definition 1.4 is a linearly ordered field.

Proposition 1.6. The structure $({}^*\mathbb{R},+,\cdot,<)$ is a linearly ordered field.

Proof. In order to show that $^*\mathbb{R}$ is a field, we need only prove that $^*\mathbb{R}$ contains inverses since the associative, commutative, and distributive laws as well as closure are inherited from the real numbers. Let $[\langle a_n\rangle]\in{}^*\mathbb{R}$ such that $[\langle a_n\rangle]\neq[\langle 0,0,0,\ldots\rangle]$. Then the set $\{n\in\mathbb{N}\mid a_n=0\}\notin\mathcal{U}$. Thus we know that the complement of this set is in $\mathcal{U}$. So define $\langle b_n\rangle$ by

$$b_n=\begin{cases}0, & \text{if } a_n=0;\\ 1/a_n, & \text{if } a_n\neq 0.\end{cases}$$

Then the product $[\langle a_n\rangle][\langle b_n\rangle]$ is equivalent to 1 since $\{n\in\mathbb{N}\mid a_nb_n=1\}=\{n\in\mathbb{N}\mid a_n\neq 0\}\in\mathcal{U}$. Therefore a_n is invertible and $^*\mathbb{R}$ is a field.

Now let $[\langle a_n\rangle],[\langle b_n\rangle]\in{}^*\mathbb{R}$ such that $[\langle a_n\rangle]\neq[\langle b_n\rangle]$. To prove that $^*\mathbb{R}$ is linearly ordered, we must show that either $[\langle a_n\rangle]<[\langle b_n\rangle]$ or $[\langle b_n\rangle]<[\langle a_n\rangle]$. Let $A=\{n\in\mathbb{N}\mid a_n<b_n\}$, $B=\{n\in\mathbb{N}\mid b_n<a_n\}$, and $E=\{n\in\mathbb{N}\mid a_n=b_n\}$. Since

$[\langle a_n\rangle] \neq [\langle b_n\rangle]$, $E \notin \mathcal{U}$. So the complement of *E*, which is $A \cup B$, is in $\mathcal{U}$. If $A \in \mathcal{U}$, then $[\langle a_n\rangle] < [\langle b_n\rangle]$. If $A \notin \mathcal{U}$, then $B \cup E \in \mathcal{U}$, in which case $(A \cup B) \cap (B \cup E) = B \in \mathcal{U}$, so $[\langle b_n\rangle] < [\langle a_n\rangle]$. Thus $[\langle a_n\rangle]$ and $[\langle b_n\rangle]$ are ordered.

Now that we have shown that $^*\mathbb{R}$ is a linearly ordered field, we wish to show that $^*\mathbb{R}$ has a proper subfield which is isomorphic to $\mathbb{R}$. We embed the reals in the hyperreals by defining the map $\theta : \mathbb{R} \to {}^*\mathbb{R}$ by

$$\theta(r) = [\langle r, r, r, \ldots\rangle].$$

To show that θ maps $\mathbb{R}$ to a proper subfield of $^*\mathbb{R}$, note that $[\langle 1, 2, 3, \ldots\rangle]$ is not equivalent to the image of any real number. This equivalence class is actually an example of an infinite number as is defined below.

1.2 Infinite and Infinitesimal Numbers

Whether a number is infinite or infinitesimal is independent of the number's sign, so the definitions of infinite and infinitesimal numbers will involve absolute values. Absolute value is a function that is already defined on the real numbers which we need to extend to the hyperreal numbers. To do this, we simply apply the absolute value entrywise to an equivalence class representative in $^*\mathbb{R}$. Much more will be said about extending functions from the real numbers to the hyperreal numbers shortly, but right now we only need the following definition.

Definition 1.7. For all $[\langle a_n\rangle] \in {}^*\mathbb{R}$, $|[\langle a_n\rangle]| = [\langle |a_n| \rangle]$.

Note that this definition is equivalent to the hyperreal analog of the definition of absolute value on the reals,

$$|[\langle a_n\rangle]| = \begin{cases} [\langle a_n\rangle], & \text{if } [\langle a_n\rangle] \geq 0; \\ -[\langle a_n\rangle], & \text{if } [\langle a_n\rangle] < 0. \end{cases}$$

This is because if $[\langle a_n\rangle] \geq 0$ then $a_n = |a_n|$ on the same set in the ultrafilter for which $a_n \geq 0$, and if $[\langle a_n\rangle] < 0$ then $-a_n = |a_n|$ on the same set in the ultrafilter for which $a_n < 0$.

From this point on, we will use single letters to denote elements of $^*\mathbb{R}$. When we speak of a *real* number $r \in {}^*\mathbb{R}$, we mean the equivalence class $[\langle r, r, r, \ldots\rangle]$. Although technically speaking, the sets $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ are not true subsets of the hyperreals, we will use these symbols to refer to the images of these sets embedded in $^*\mathbb{R}$.

We can now define what is means for a number to be infinite or infinitesimal.

Definition 1.8. Let $a \in {}^*\mathbb{R}$. Then a is *infinite* if and only if $|a| > r$ for every positive real number *r*. If a is not infinite then it is finite. We call *a infinitesimal* if and only if $|a| < r$ for every positive real number *r*.

Note that the set of finite numbers is a subring of $^*\mathbb{R}$ and the set of infinitesimals is an ideal of the finite numbers. More importantly, note that the set of hyperreal numbers is nonarchimedean. Not every bounded subset of $^*\mathbb{R}$ is guaranteed to have a least upper bound or greatest lower bound. For example, the set of infinite numbers is bounded below by any finite number but has no greatest lower bound.

We now introduce an equivalence relation which associates numbers which only differ by an infinitesimal.

Definition 1.9. Given $x, y \in {}^*\mathbb{R}$, *x* and *y* are *near* or *infinitely close* if and only if $x - y$ is an infinitesimal. In this case, we write $x \simeq y$. The equivalence classes induced by $\simeq$ are called *monads*. Thus, the monad about a number $a \in {}^*\mathbb{R}$, written $m(a)$ is defined by $m(a) = \{x \in {}^*\mathbb{R} \mid x \simeq a\}$.

$^*\mathbb{R}$ is also partitioned into classes whose elements differ by finite amounts.

Definition 1.10. The *galaxy* about a number $a \in {}^*\mathbb{R}$, written $G(a)$, is defined by $G(a) = \{x \in {}^*\mathbb{R} \mid x - a \text{ is finite}\}$.

Using monads and galaxies, we can use $m(0)$ to represent the set of all infinitesimal numbers and $G(0)$ to represent the set of all finite numbers. As shown below, the set of real numbers is isomorphic to the quotient ring $G(0)/m(0)$.

Numbers in $^*\mathbb{R}$ which are not images of real numbers are called *nonstandard*. Nonstandard finite numbers are always infinitely close to exactly one real number, which is called its standard part.

Proposition 1.11. Let a be a finite hyperreal number. Then there exists a unique real number *r* such that $r \simeq a$.

Proof. Let $A = \{x \in \mathbb{R} \mid x \leq a\}$. Since a is finite, A is nonempty and is bounded above. Let r be the least upper bound of A. For any real $\varepsilon > 0$, $r - \varepsilon \in A$ and $r + \varepsilon \notin A$ and thus $r - \varepsilon \leq a < r + \varepsilon$. So $|r - a| \leq \varepsilon$ from which is follows that $r \simeq a$. To show that this r is unique, suppose that there exists a real number s such that $s \simeq a$. Then since $\simeq$ is transitive, $s \simeq r$. So $|s-r|<\varepsilon$ for every real $\varepsilon > 0$, and thus $s = r$.

Definition 1.12. Let $a \in G(0)$. Then the *standard part* of a, denoted by $\mathrm{st}(a)$ or $°a$, is the real number r such that $a \simeq r$. The function st is called the *standard part map*.

The standard part map is easily shown to be an order preserving homomorphism from $G(0)$ onto $\mathbb{R}$ with kernel $m(0)$. The quantity $x - \mathrm{st}(x)$ is sometimes called the nonstandard part of x.

1.3 Relations and *-Transforms

We now discuss the method through which functions defined on the set of real numbers are extended to the hyperreals. The method actually applies to any arbitrary relation defined on $\mathbb{R}$, the set of which includes all of the functions defined on $\mathbb{R}$. The process of extending a relation from $\mathbb{R}$ to $^*\mathbb{R}$ is called a *-transform for which the general definition is next given.

Definition 1.13. Let P be an n-ary relation on $\mathbb{R}$. Then the *-transform of P, denoted by $^*P$ is the set of all n-tuples $([\langle (a_1)_i \rangle], [\langle (a_2)_i \rangle], \ldots, [\langle (a_n)_i \rangle]) \in (^*\mathbb{R})^n$ satisfying $\{i \in \mathbb{N} \mid ((a_1)_i, (a_2)_i, \ldots, (a_n)_i) \in P\} \in \mathcal{U}$.

An n-ary relation on $\mathbb{R}$ is simply a subset of $\mathbb{R}^n$. Thus, unary relations on $\mathbb{R}$ are just subsets of $\mathbb{R}$. Equality, the ordering given by <, and functions of one variable are examples of binary relations. In order to acquire a more intuitive feeling for how relations on $\mathbb{R}$ are extended to $^*\mathbb{R}$, we consider a few examples.

Let $A \subseteq \mathbb{R}$. Then A is a unary relation on $\mathbb{R}$, so $^*A$ consists of those elements $[\langle a_n \rangle] \in {}^*\mathbb{R}$ such that $\{n \in \mathbb{N} \mid a_n \in A\} \in \mathcal{U}$ (i.e., $a_n \in A$ almost everywhere). It should now be clear why the notation $^*\mathbb{R}$ is used to represent the set of hyperreal numbers, for if $A = \mathbb{R}$ then $\{n \in \mathbb{N} \mid a_n \in A\} = \mathbb{N} \in \mathcal{U}$, so $^*\mathbb{R}$ consists of all equivalence classes represented by any sequence of real numbers.

The set $^*\mathbb{N}$ is called the set of hypernatural numbers and consists of the numbers $[\langle a_n \rangle] \in {}^*\mathbb{R}$ for which $a_n \in \mathbb{N}$ almost everywhere. Likewise, $^*\mathbb{Z} = \{[\langle a_n \rangle] \in {}^*\mathbb{R} \mid a_n \in \mathbb{Z} \text{ a.e.}\}$ and $^*\mathbb{Q} = \{[\langle a_n \rangle] \in {}^*\mathbb{R} \mid a_n \in \mathbb{Q} \text{ a.e.}\}$. These are called the hyperintegers and hyperrationals respectively.

Proposition 1.14. Let $S \subseteq \mathbb{R}$ be a set which contains an infinite subset of $\mathbb{N}$. Then $^*S$ contains infinite numbers.

Proof. Let A be an infinite subset of $\mathbb{N}$ contained in S. We construct an infinite number in $^*S$ as follows. Let a_1 be the least element of A and then choose each a_n to be the least element of the set $A-\{a_1,a_2,\dots,a_{n-1}\}$. Since A has no upper bound, the sequence $\langle a_n\rangle$ must have the property that given any positive real number r, there exists an $N\in\mathbb{N}$ such that for all $n>N$, we have $a_n>r$. Thus $a_n>r$ almost everywhere, satisfying the requirement for $[\langle a_n\rangle]\in {}^*S$ to be infinite.

Given a set $S\subseteq {}^*\mathbb{R}$, we use the notation S_∞ to represent the set of infinite numbers in S. Thus, $^*\mathbb{Z}_\infty$ is the set of infinite hyperintegers in $^*\mathbb{R}$.

Let us consider equality on $\mathbb{R}$ for a moment as a set of duplets and write $(a,b)\in E$ if and only if $a=b$. Then $^*E$ is the set of all duplets $([\langle a_n\rangle],[\langle b_n\rangle])\in({}^*\mathbb{R})^2$ satisfying $\{n\in\mathbb{N}\mid a_n=b_n\}\in\mathcal{U}$, which is exactly how we defined equality on the hyperreals.

A function f on $\mathbb{R}$ of n variables can be thought of as a set of $(n+1)$-tuples having the property that if $(c_1,c_2,\dots,c_n,a)\in f$ and $(c_1,c_2,\dots,c_n,b)\in f$, then $a=b$. The *-transform of f is the set of all $(n+1)$-tuples of the form $([\langle(c_1)_i\rangle],[\langle(c_2)_i\rangle],\dots,[\langle(c_n)_i\rangle],[\langle a_n\rangle])\in({}^*\mathbb{R})^{n+1}$ satisfying

$$\{i\in\mathbb{N}\mid((c_1)_i,(c_2)_i,\dots,(c_n)_i,a_i)\in f\}\in\mathcal{U}.$$

So if f is a function of one variable, then $^*f([\langle c_i\rangle])=[\langle f(c_i)\rangle]$. That is, we just let f operate entrywise on an equivalence class representative of a number in $^*\mathbb{R}$.

Note that operations such as addition and multiplication are actually functions of two variables. In the next section, we will be taking logarithms of hyperreal numbers using a hyperreal number for the base. The logarithm is also a function of two variables so we have

$$^*\log_{[\langle b_n\rangle]}[\langle a_n\rangle]=\left[\left\langle\log_{b_n}a_n\right\rangle\right].$$

If we choose $[\langle b_n\rangle]=[\langle 1,2,3,\dots\rangle]$ and $[\langle a_n\rangle]=[\langle 2,2,2,\dots\rangle]$, then the first entry of $^*\log_{[\langle b_n\rangle]}[\langle a_n\rangle]$ is $\log_1 2$, which is undefined. This is acceptable, however, since we can assign any value we wish to the first entry and the function will still hold true on a set in the ultrafilter $\mathcal{U}$. As long as the *-transform of a function is defined almost everywhere for the entries of an equivalence class representative $\langle a_n\rangle$, it is defined for $[\langle a_n\rangle]$.

1.4 Sequences and Series

Sequences and series of hyperreal numbers will be an important area of study later in this paper. When we speak of a sequence of hyperreal numbers, we are actually talking about a sequence of sequences of real numbers, and this sequence is indexed not by the natural numbers, but by the hypernatural numbers. Below, we examine how sequences of real numbers indexed by the natural numbers are *-transformed to sequences of hyperreal numbers indexed by the hypernatural numbers.

Let $\langle s_n \mid n \in \mathbb{N} \rangle$ be a sequence of real numbers. This sequence is actually a function $s : \mathbb{N} \to \mathbb{R}$ where $s(n) = s_n$, so we can think of $\langle s_n \rangle$ as the set of duplets $\{(1, s_1), (2, s_2), (3, s_3), \ldots\}$. The *-transform of s is the function $^*s : {}^*\mathbb{N} \to {}^*\mathbb{R}$ where $^*s([\langle a_n \rangle]) = [\langle s(a_n) \rangle]$ for any $[\langle a_n \rangle] \in {}^*\mathbb{N}$. The sequence $\langle {}^*s_n \mid n \in {}^*\mathbb{N} \rangle$ is an extension of the sequence $\langle s_n \mid n \in \mathbb{N} \rangle$ in that for any $n \in \mathbb{N}$, $^*s_n$ is simply the image of s_n in the hyperreals.

If a sequence $\langle s_n \rangle$ tends to a limit in the real numbers, then the sequence $\langle {}^*s_n \rangle$ tends to the same limit in the hyperreal numbers. This is proven shortly, but first we need to discuss a little notation. The symbol ∞ is used in the real number system to denote that which is potentially arbitrarily large. Thus the expression $\lim_{n \to \infty} s_n$ denotes the value that s_n approaches as n becomes an arbitrarily large natural number. In the hyperreal number system, the symbol ∞ has the same meaning, but by arbitrarily large, we mean even larger than any infinite number in $^*\mathbb{R}$. So the expression $\lim_{n \to \infty} {}^*s_n$ denotes the value that $^*s_n$ approaches as n becomes an arbitrarily large *hypernatural* number. We now have the following proposition.

Proposition 1.15. Suppose that $\lim_{n \to \infty} s_n = L$ for some real number L. Then $\lim_{n \to \infty} {}^*s_n = L$.

Proof. The fact that $\lim_{n \to \infty} s_n = L$ means that given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - L| < \varepsilon$. Now let ε be a positive hyperreal number and let $\langle \varepsilon_i \rangle$ be an equivalence class representative of ε. We define the hypernatural number $N = [\langle N_i \rangle]$ by choosing N_i to be any natural number for which $|s_n - L| < \varepsilon_i$ for all $n > N_i$. Then for any hypernatural number $n > N$, $|{}^*s_n - L| < \varepsilon$. So $\lim_{n \to \infty} {}^*s_n = L$.

More important to this paper are infinite series of hyperreal numbers. We define an infinite series in terms of its sequence of partial sums. Consider the standard sequence

$$s(n,x_1,x_2,\dots,x_m)=\sum_{k=1}^{n}a(k,x_1,x_2,\dots,x_m),$$

which is defined as a function of the natural index n and real numbers $x_1,x_2,\dots,x_m$. The sum of the infinite series is represented by $\lim_{n\to\infty} s(n,x_1,x_2,\dots,x_m)$. The *-transform of s is the sequence

$$*s(n,x_1,x_2,\dots,x_m)=*\sum_{k=1}^{n}*a(k,x_1,x_2,\dots,x_m),$$

where $x_1,x_2,\dots,x_m$ are hyperreal numbers and the right hand side represents the definition of the nonstandard summation from 1 to a hypernatural number n. This summation is written in equivalence class form as

$$*\sum_{k=1}^{n}*a(k,x_1,x_2,\dots,x_m)=\left[\left\langle\sum_{k=1}^{n_i}a\left(k,(x_1)_i,(x_2)_i,\dots,(x_m)_i\right)\right\rangle\right].$$

An important addition to this definition is the requirement that for each of the hyperreal numbers $x_1,x_2,\dots,x_m$, only one equivalence class representative may be used throughout the entire summation. That is, to calculate $s(n,x_1,x_2,\dots,x_m)$ for a single hypernatural number n, we first choose equivalence class representatives for each of the $x_1,x_2,\dots,x_m$ and use the same representatives each time that we evaluate $*a(k,x_1,x_2,\dots,x_m)$ in the summation. Without this restriction, is it possible to find ill-defined summations when n is infinite by choosing different equivalence class representatives for each index k.

The sum of the infinite series

$$*\sum_{k=1}^{n}*a(k,x_1,x_2,\dots,x_m)$$

is defined to be $\lim_{n\to\infty} *s(n,x_1,x_2,\dots,x_m)$ where n approaches infinity through the hypernatural numbers. If this limit exists and is equal to any hyperreal number, including infinite numbers, then the series is said to be convergent.

We consider an example that is used later in this paper. Let $s(n,a,b,t)$ represent the sequence of partial sums of the binomial series for $(a+b)^t$. This can be written as the summation

$$s(n,a,b,t)=\sum_{k=1}^{n}\binom{t}{k}a^{t-k}b^{k}.$$

In order to use the binomial expansion in the case where a, b, and t are hyperreal numbers, we need the *-transform of s, which is given by

$$\begin{aligned} {}^*s(n,a,b,t) &= {}^*\sum_{k=1}^{n} \binom{t}{k} a^{t-k} b^{k} \\ &= \left[\left\langle \sum_{k=1}^{n_i} \binom{t_i}{k} a_i^{t_i-k} b_i^{k} \right\rangle \right]. \end{aligned}$$

In this summation, we must use fixed pre-chosen equivalence class representatives for a, b, and t for every index k.

1.5 Limits

One final result of nonstandard analysis that is needed in this paper is the following statement about the limit of a function at a point or at infinity.

Proposition 1.16. Let L be a real number.

(a) $\lim_{x\to a} f(x) = L$ if and only if ${}^*f(x) \simeq L$ for all $x \in m(a) - \{a\}$.

(b) $\lim_{x\to \infty} f(x) = L$if and only if ${}^*f(x) \simeq L$ for all positive infinite x.

Proof. All parts are proven by contrapositive.

(a) Suppose ${}^*f(z) \not\simeq L$ for some $z \simeq a$. We want to show that there exists an $\varepsilon > 0$ such that for any $\delta > 0$ there exists an x such that $|x-a| < \delta$, but $|f(x) - L| \geq \varepsilon$. Since ${}^*f(z) \not\simeq L$, we have $|{}^*f(z) - L| \gtrsim 0$. If ${}^*f(z)$ is finite, choose $\varepsilon = \mathrm{st}\left(\frac{1}{2}|{}^*f(z) - L|\right)$, and if ${}^*f(z)$ is infinite, choose ε to be any positive real number. Let $\langle z_n \rangle$ be an equivalence class representative of z. Since $z \simeq a$, we know that

$$Z = \{n \in \mathbb{N} \mid |z_n - a| < \delta\} \in \mathcal{U}.$$

We also know that

$$F = \{n \in \mathbb{N} \mid |f(z_n) - L| \geq \varepsilon\} \in \mathcal{U}.$$

Choose $N \in Z \cap F$ and let $x = z_N$. Then $|x-a| < \delta$, but $|f(x) - L| \geq \varepsilon$. Therefore $\lim_{x\to a} f(x) \neq L$.

Now we assume $\lim_{x\to a} f(x) \neq L$. Then there exists an $\varepsilon > 0$ such that for every $\delta > 0$ we can find x such that $|x-a| < \delta$, but $|f(x) - L| \geq \varepsilon$. We construct a sequence $\langle x_n \rangle$ by choosing each x_n such that $|x_n - a| < 1/n$ and $|f(x_n) - L| \geq \varepsilon$. Then we have $[\langle x_n \rangle] \simeq a$ since for any positive real number r, $|x_n - a| < r$ for all $n > N$ where N is the least natural number satisfying $1/N < r$. But $*f([\langle x_n \rangle]) \neq L$ because $|f(x_n) - L| \geq \varepsilon$ for all $n \in \mathbb{N}$. So we have found a number infinitely close to a for which the function value is not infinitely close to L.

(b) Suppose $*f(z) \neq L$ for some positive infinite number z. We want to show that there exists an $\varepsilon > 0$ such that for any $m \in \mathbb{N}$ there exists an $x > m$ such that $|f(x) - L| \geq \varepsilon$. As in part (a), $|*f(z) - L| \gtrsim 0$. So if $*f(z)$ is finite, choose $\varepsilon = \mathrm{st}\left(\frac{1}{2}|*f(z) - L|\right)$, and if $*f(z)$ is infinite, choose ε to be any positive real number. Let $\langle z_n \rangle$ be an equivalence class representative of z. Since z is infinite, we know that

$$Z = \{n \in \mathbb{N} \mid z_n > m\} \in \mathcal{U}.$$

We also know that

$$F = \{n \in \mathbb{N} \mid |f(z_n) - L| \geq \varepsilon\} \in \mathcal{U}.$$

Choose $N \in Z \cap F$ and let $x = z_N$. Then $x > m$, but $|f(x) - L| \geq \varepsilon$. Therefore $\lim_{x\to\infty} f(x) \neq L$.

Now we assume $\lim_{x\to\infty} f(x) \neq L$. Then there exists an $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ we can find $x > N$ such that $|f(x) - L| \geq \varepsilon$. We construct a sequence $\langle x_n \rangle$ by choosing each x_n such that $x_n > n$ and $|f(x_n) - L| \geq \varepsilon$. Then $[\langle x_n \rangle]$ is infinite since for any positive real number r, $x_n > r$ for all $n > r$. But $*f([\langle x_n \rangle]) \neq L$ because $|f(x_n) - L| \geq \varepsilon$ for all $n \in \mathbb{N}$. So we have found an infinite number for which the function value is not infinitely close to L.

2 Development of New Hyperreal Structures

This section introduces the concept of superiority and the equivalence relation called local equality. Many structures which arise in the set of hyperreal numbers due to these concepts are examined along with their relationships to existing structures. Succeeding sections discuss applications of the properties of these new structures.

The following notation is used in this and later sections. The set of all infinite numbers in $^*\mathbb{R}$ will be denoted by **T**, and the set of all infinitesimal numbers in $^*\mathbb{R}$ will be denoted by **S**. The symbols $\lesssim$ and $\gtrsim$ will be used to mean "less than or infinitely close to" and "greater than or infinitely close to" respectively. The symbols $\lnsim$ and $\gnsim$ will be used to mean "less than but *not* infinitely close to" and "greater than but *not* infinitely close to" respectively.

2.1 Orders of Numbers and the Superiority Relation

The order of a number and all subsequently dependent relations are based upon the choice of a positive infinite number ω. We do not give an explicit equivalence class representative for ω since all calculations that we perform which involve ω do not actually rely on its value. Thus all of the relations and structures defined in this section have the same properties for every possible choice of ω. We need only remember that during any one calculation that ω represents a constant positive infinite hyperreal number. This method allows us to perform computations in the hyperreal numbers and at the same time abstract beyond the need to know anything about the ultrafilter used to construct the hyperreals or the sequences that represent equivalence classes. All symbolic manipulation is done with real numbers and functions of ω. We use the letter α to represent $1/\omega$ for any choice of ω. The arbitrary nature of ω and α allows us to show in a single calculation that certain properties hold for all positive infinite or infinitesimal numbers.

We first define an equivalence relation that partitions $^*\mathbb{R}$ into what are called zones. This is done by considering the logarithm base ω of each hyperreal number. We will use the symbol log to represent the function $^*\log$ on $^*\mathbb{R}$ as well as the function log on $\mathbb{R}$.

Definition 2.1. The *order* of a number $a \in {}^*\mathbb{R}$, written $\operatorname{ord}(a)$, is defined as $\operatorname{ord}(a) = \log_\omega |a|$. The order of 0 is defined to be $-\infty$.

The order of a number represents the size of the number on a scale which transcends infinitesimal, finite, and infinite numbers. Although this order depends upon the choice of ω, we have the following property that does not depend on the value of ω, but only on the fact that ω is infinite.

Lemma 2.2. All numbers in $G(0) - m(0)$ (i.e., all finite non-infinitesimals) have infinitesimal order.

Proof. Let $x \in G(0) - m(0)$. Since x is not infinitesimal or infinite, there exist positive real numbers a and b such that $a < |x| < b$. For every positive real number r, $\operatorname{ord}(r) = \ln r / \ln \omega \in \mathbf{S}$. Since the logarithm is a strictly increasing function, we have $\operatorname{ord}(a) < \operatorname{ord}(x) < \operatorname{ord}(b)$. So x has infinitesimal order.

Clearly, every infinite number has positive order and every infinitesimal number has negative order. What is interesting is that although most infinite and infinitesimal numbers do not have infinitesimal order, there are elements of $\mathbf{T}$ and $\mathbf{S}$ that do, and these numbers are given the following special names.

Definition 2.3. If $t \in \mathbf{T}$ and $\operatorname{ord}(t) \in \mathbf{S}$, then t is called *semi-infinite*. The set of infinite numbers less the set of semi-infinite numbers is denoted by $\mathbf{T}'$.

Definition 2.4. If $s \in \mathbf{S}$ and $\operatorname{ord}(s) \in \mathbf{S}$, then s is called *semi-infinitesimal*. The set of infinitesimal numbers less the set of semi-infinitesimal numbers is denoted by $\mathbf{S}'$.

An example of a semi-infinite number is $\ln \omega$ whose order is $\ln \ln \omega / \ln \omega$. All semi-infinitesimal numbers are reciprocals of semi-infinite numbers and vice-versa, so $1/\ln \omega$ is semi-infinitesimal. Properties of semi-infinite and semi-infinitesimal numbers are identified throughout this section.

The order of each hyperreal number is used to define the following relation on ${}^*\mathbb{R}$.

Definition 2.5. Given $a, b \in {}^*\mathbb{R}$, we say a and b are *isometric* and write $a \mathrel{\widehat{=}} b$ if and only if $\operatorname{ord}(a) \simeq \operatorname{ord}(b)$.

The relation $\widehat{=}$ is an equivalence relation since $\mathbf{S}$ is a group under addition. The name "isometric" is used because two numbers whose orders differ only by an infinitesimal are of the same general size when considered as members of the vast infinitesimal and infinite extent of the set of hyperreal numbers.

We now introduce the superiority relation that is used to relate two numbers which are not isometric.

Definition 2.6. Let $a,b \in {}^*\mathbb{R}$ such that $a \not\mathrel{\widehat{=}} b$. If $\operatorname{ord}(a) \lnapprox \operatorname{ord}(b)$, then we say a is *inferior* to b and write $a \mathrel{\widehat{<}} b$, and if $\operatorname{ord}(a) \gnapprox \operatorname{ord}(b)$, then we say a is *superior* to b and write $a \mathrel{\widehat{>}} b$.

The symbol $\widehat{\leq}$ is used to mean "inferior to or isometric to", and the symbol $\widehat{\geq}$ is used to mean "superior to or isometric to". Note that $0 \mathrel{\widehat{<}} a$ for all nonzero $a \in {}^*\mathbb{R}$. Also note that if $|a| \leq b$ then $a \mathrel{\widehat{\leq}} b$. This property will be useful in several proofs later in this paper.

If two numbers a and b in ${}^*\mathbb{R}$ are related by $\widehat{=}$, $\widehat{<}$, or $\widehat{>}$, then multiplication of both by any nonzero $x \in {}^*\mathbb{R}$ preserves this relation.

Lemma 2.7. Let $a,b,x \in {}^*\mathbb{R}$. Then $a \mathrel{\widehat{=}} b$ implies $ax \mathrel{\widehat{=}} bx$, and if $x \neq 0$ then $a \mathrel{\widehat{<}} b$ implies $ax \mathrel{\widehat{<}} bx$.

Proof. Suppose $a \mathrel{\widehat{=}} b$. Then $\operatorname{ord}(a) \simeq \operatorname{ord}(b)$. Adding $\operatorname{ord}(x)$ to both sides, we have $\operatorname{ord}(a) + \operatorname{ord}(x) \simeq \operatorname{ord}(b) + \operatorname{ord}(x)$. Therefore $\operatorname{ord}(ax) \simeq \operatorname{ord}(bx)$, so $ax \mathrel{\widehat{=}} bx$.

Now suppose $a \mathrel{\widehat{<}} b$. Then $\operatorname{ord}(a) \lnapprox \operatorname{ord}(b)$, so by a similar argument $\operatorname{ord}(ax) \lnapprox \operatorname{ord}(bx)$ and thus $ax \mathrel{\widehat{<}} bx$.

Addition of equals does not preserve the relations $\widehat{=}$, $\widehat{<}$, and $\widehat{>}$. Consider as an example that $1+\alpha \mathrel{\widehat{=}} 1+\alpha^2$, but if we add -1 to both sides then we would have $a \mathrel{\widehat{=}} a^2$, which is not true.

2.2 Zones and Worlds

An equivalence class created by the equivalence relation $\widehat{=}$ is given the following name.

Definition 2.8. The equivalence class induced by $\widehat{=}$ containing $a \in {}^*\mathbb{R}$ is called the *zone* about a. This is written $\operatorname{zone}(a) = \{x \in {}^*\mathbb{R} \mid x \mathrel{\widehat{=}} a\}$. We define $\operatorname{zone}(0) = \{0\}$.

The zone about a is the set of all numbers whose order is in the monad about $\operatorname{ord}(a)$. Thus it is possible to express the zone about a as

$$\text{zone}(a) = \omega^{m(\text{ord}(a))} \cup -\omega^{m(\text{ord}(a))}.$$

Note that since $\text{ord}(x) = \text{ord}(-x)$ for all $x \in {}^*\mathbb{R}$, if $x \in \text{zone}(a)$ then $-x \in \text{zone}(a)$. Also note that since $\text{ord}(1) = 0$, zone(1) is the set of all elements of ${}^*\mathbb{R}$ having infinitesimal order. Therefore zone(1) contains all of the nonzero real numbers.

Theorem 2.9. zone(1) is a multiplicative subgroup of ${}^*\mathbb{R}$.

Proof. We need to prove closure and containment of inverses. Let $a, b \in \text{zone}(1)$. Then $\text{ord}(a) \in \mathbf{S}$ and $\text{ord}(b) \in \mathbf{S}$. Therefore, $\text{ord}(ab) = \text{ord}(a) + \text{ord}(b) \in \mathbf{S}$. So $ab \in \text{zone}(1)$ and thus zone(1) is closed under multiplication. zone(1) contains multiplicative inverses since for all $x \in {}^*\mathbb{R}$, $\text{ord}(x^{-1}) = -\text{ord}(x)$ and **S** contains additive inverses.

Every zone is a coset of zone(1). We therefore have the properties that for all $a \in {}^*\mathbb{R}$, $\text{zone}(a) = a\,\text{zone}(1)$ and zone(a) is closed under multiplication by elements of zone(1).

All semi-infinite and semi-infinitesimal numbers are contained in zone(1). We can therefore express the set of semi-infinite numbers as $\mathbf{T} \cap \text{zone}(1)$, and we can express the set of semi-infinitesimal numbers as $\mathbf{S} \cap \text{zone}(1)$.

No zone (excluding zone(0)) is closed under addition since for any $b \mathrel{\hat{<}} a$, $a + b \in \text{zone}(a)$, but $a + b - a \notin \text{zone}(a)$. The smallest additive subgroup of ${}^*\mathbb{R}$ containing zone(a) therefore contains every number which is inferior to *a*. This set is given the following name.

Definition 2.10. The *world* about $a \in {}^*\mathbb{R}$, denoted $W(a)$, is defined by $W(a) = \{x \in {}^*\mathbb{R} \mid x \mathrel{\hat{\leq}} a\}$.

We use the notation $W_0(a)$ to represent the set $\{x \in {}^*\mathbb{R} \mid x \mathrel{\hat{<}} a\}$. $W_0(a)$ can be thought of as the interior of the set of zones contained in $W(a)$, and zone(a) can be thought of as the boundary of the set of zones contained in $W(a)$. We thus have for the zone about *a* the alternate notation $\partial W(a)$.

Theorem 2.11. $W(a)$ and $W_0(a)$ are groups under addition for all $a \in {}^*\mathbb{R}$.

Proof. $0 \in W(a)$ and $0 \in W_0(a)$ because $0 \mathrel{\hat{\leq}} a$ for all $a \in {}^*\mathbb{R}$. Each zone which is contained in $W(a)$ contains its own additive inverses. We are left with proving closure, which we first prove for the $W(a)$ case.

Let $x, y \in W(a)$. Without loss of generality, we can assume that $x \hat{\geq} y$. We will show that $x + y \in W(x) \subseteq W(a)$. This means we must show that $\operatorname{ord}(x+y) \lesssim \operatorname{ord}(x)$. Let $z = \max(|x|,|y|)$. Then

$$\begin{aligned}
\operatorname{ord}(x+y) &= \log_\omega |x+y| \\
&\leq \log_\omega (|x|+|y|) \\
&\leq \log_\omega 2z \\
&= \log_\omega 2 + \log_\omega z \\
&\simeq \operatorname{ord}(z). \qquad (2.1)
\end{aligned}$$

If $x \hat{=} y$ then $z \hat{=} x$. If $x \hat{>} y$ then $z = |x|$. In both cases, $\operatorname{ord}(z) \simeq \operatorname{ord}(x)$. Therefore, from the above equation, $\operatorname{ord}(x+y) \lesssim \operatorname{ord}(x)$.

Closure of $W_0(a)$ is proven in the exact same manner since for $x, y \in W_0(a)$ with $x \hat{\geq} y$, $W(x) \subset W_0(a)$.

An important observation is that if $x \hat{>} y$ then $x + y \hat{=} x$ since otherwise, if $x + y = z \hat{<} x$ then $x = z - y \hat{<} x$, a contradiction. This is the fundamental concept behind what we soon define to be local equality.

Theorem 2.12. $W(a)$ and $W_0(a)$ are rings if and only if $a \hat{\leq} 1$.

Proof. Assume $a \hat{\leq} 1$. By Theorem 2.11, $W(a)$ and $W_0(a)$ are additive subgroups of $^*\mathbb{R}$. We need only prove closure under multiplication. Let $x, y \in W(a)$. Since $a \hat{\leq} 1$ and both $x \hat{\leq} a$ and $y \hat{\leq} a$, $\operatorname{ord}(x) \lesssim 0$ and $\operatorname{ord}(y) \lesssim 0$. Therefore, $\operatorname{ord}(xy) = \operatorname{ord}(x) + \operatorname{ord}(y) \lesssim 0$. So $xy \in W(a)$ and $W(a)$ is thus closed under multiplication. For $x, y \in W_0(a)$, we have the simpler situation that $\operatorname{ord}(x) < 0$ and $\operatorname{ord}(y) < 0$. Therefore, $\operatorname{ord}(xy) < 0$ and we have $xy \in W_0(a)$.

Now suppose that $a \hat{>} 1$. Then $\operatorname{ord}(a) \gtrsim 1$. So we have $\operatorname{ord}(a^2) = 2\operatorname{ord}(a) \gtrsim \operatorname{ord}(a)$ and thus $a^2 \notin W(a)$. Therefore $W(a)$ is not closed under multiplication. Also, $W_0(a)$ is not closed since if we let $b = \sqrt{|a|} \in W_0(a)$, then $b^2 \notin W_0(a)$.

It turns out that for any two rings of the type mentioned in Theorem 2.12, the smaller subring is an ideal of the larger.

Theorem 2.13. Let $a, b \in {}^*\mathbb{R}$ such that $a \hat{<} b \hat{\leq} 1$. Then

(a) $W(a) \triangleleft W(b)$

(b) $W_0(a) \lhd W(b)$

(c) $W(a) \lhd W_0(b)$

(d) $W_0(a) \lhd W_0(b)$

Proof.

(a) Let $x \in W(a)$ and $y \in W(b)$. Then $\operatorname{ord}(x) \lnsim 0$ (since $a \mathrel{\widehat{<}} 1$) and $\operatorname{ord}(y) \lesssim 0$. So $\operatorname{ord}(xy) = \operatorname{ord}(x) + \operatorname{ord}(y) \lesssim \operatorname{ord}(x)$, which implies $xy \mathrel{\widehat{\le}} x$, and thus $xy \in W(a)$.

(b) Identical to (a).

(c) Similar to (a), except $\operatorname{ord}(y) \lnsim 0$. So $\operatorname{ord}(xy) \lnsim \operatorname{ord}(x)$, which implies $xy \mathrel{\widehat{<}} x$.

(d) Identical to (c).

Theorem 2.14. $W_0(1)$ is a maximal ideal of $W(1)$.

Proof. Suppose there exists an ideal I of $W(1)$ with $W_0(1) \subset I \subset W(1)$. Then there exists $x \in I$ such that $x \mathrel{\widehat{=}} 1$ and is therefore an element of $\operatorname{zone}(1)$. But by Theorem 2.9, $\operatorname{zone}(1)$ contains multiplicative inverses, so x is a unit and thus $I = W(1)$. So $W_0(1)$ must be maximal.

We have now constructed a field $W(1)/W_0(1)$ considerably smaller than $^*\mathbb{R}$ that still contains an isomorphic copy of the real numbers through the map $r \mapsto [\langle r, r, r, \ldots \rangle] + W_0(1)$ and also contains infinite and infinitesimal numbers such as $\ln \omega + W_0(1)$ and $1/\ln \omega + W_0(1)$.

2.3 Local Equality

The concept of local equality involves choosing a "level" to work on and equating any two numbers in $^*\mathbb{R}$ that differ by an insignificant amount with respect to this level. For example, on the finite level (i.e., the level of 1), any two numbers that differ by a sufficiently small infinitesimal are considered to be locally equal. The precise definition is as follows.

Definition 2.15. Let $a, b, \ell \in {}^*\mathbb{R}$ with $\ell \neq 0$. We say a is *locally equal* to b on the level of ℓ and write $a =^{\ell} b$ if and only if $a - b \mathrel{\widehat{<}} \ell$.

Local equality on the level of ℓ is an equivalence relation since $0 \mathrel{\widehat{<}} \ell$ for all $\ell \in {}^*\mathbb{R}$ and $W_0(\ell)$ is closed under addition by Theorem 2.11. One immediate observation is that if $x \mathrel{\widehat{<}} \ell$ then $x =^{\ell} 0$. If $a =^{\ell} b$, then obviously $a + c =^{\ell} b + c$ for any $c \in {}^*\mathbb{R}$. Also, it follows immediately from Lemma 2.7 that if $a =^{\ell} b$ then $ac =^{\ell c} bc$.

If two numbers are locally equal on the level of ℓ then they are also locally equal on the level of any element of $\operatorname{zone}(\ell)$. Replacing the level representative with an isometric number does not change anything. Local equality is also preserved if the level representative is replaced by a superior number. In general, if $a =^{\ell} b$ and $m \mathrel{\widehat{\geq}} \ell$, then $a =^{m} b$.

Lemma 2.16. If $a =^{a} b$ then $a \mathrel{\widehat{=}} b$.

Proof. If $a =^{a} b$ then $a - b \mathrel{\widehat{<}} a$. So we can write $b = a + \varepsilon$ where $\varepsilon \mathrel{\widehat{<}} a$. Since $a + \varepsilon \mathrel{\widehat{=}} a$, we know $b \mathrel{\widehat{=}} a$.

An immediate consequence of this lemma is that if $a =^{a} b$ then $a =^{b} b$. This also lets us make the statement that if $a =^{a} b$ then $1/a =^{1/a} 1/b$. This is because if we divide both sides of $a =^{a} b$ by a, we have $1 =^{1} b/a$. Then dividing both sides by b gives us $1/b =^{1/b} 1/a$. By the lemma, this is equivalent to $1/a =^{1/a} 1/b$. This property of local equality is useful in Section 4.

Here are some example local equalities.

$$\alpha =^{1} \alpha^2$$

$$3 =^{\omega} 4$$

$$5 + 6\alpha =^{1} 5$$

$$7 + \frac{1}{\ln \omega} \neq^{1} 7$$

Note that in the last example given above, even though $1/\ln \omega$ is an infinitesimal, it is too large to be ignored on the finite level. The reason is that $1/\ln \omega$ is a semi-infinitesimal and semi-infinitesimals are not inferior to any finite number.

The equivalence classes of local equality on the level of 1 are analogous to monads. Numbers that are in $m(a)$ but not in $\{x \mid x =^{1} a\}$ are exactly those which differ from a by a semi-infinitesimal. So $x =^{1} y$ implies $x \simeq y$, but the converse is not necessarily true.

2.4 Summary

All of the structures and relations introduced in this section depend upon a positive infinite number ω whose exact value is left undefined but is considered to be held constant throughout any statement, proof, calculation, etc. Whenever the number ω appears in an expression, it is assumed to be the same ω that any relations appearing in the expression such as inferiority and local equality depend on through the order function.

The arbitrary nature of ω allows us to conclude statements such as if $f(\omega) \simeq L$ independently of the choice of ω, then $f(t) \simeq L$ for *all* positive infinite numbers t. This becomes an invaluable tool when used in conjunction with local equality for evaluating limits which assume an indeterminate form and also for testing infinite series for convergence. These topics are discussed in the next two sections.

3 Evaluation of Limits

We now use the concept of local equality to develop a system for evaluating certain types of limits at points where they assume indeterminate forms. The methods presented are alternatives to l'Hôpital's Rule which are generally easier to use and in most cases allow much faster computation. Several examples are given which demonstrate the new methods on limits which assume the indeterminate forms $0/0$, ∞/∞, 1^∞, and $\infty - \infty$.

3.1 Preliminary Theory

Nonstandard analysis tells us that $\lim_{x\to\infty} f(x)$ exists and is equal to L if $^*f(x) \simeq L$ for all positive infinite $x \in {}^*\mathbb{R}$ (see Section 1.5). When evaluating limits of this form, we will be able to show that $^*f(\omega) =^1 L$ independently of the choice of ω. Thus, in a single calculation, we will be able to show that $^*f(x) \simeq L$ for every infinite number ω. This implies that $\lim_{x\to\infty} f(x) = L$.

A similar method will be used for limits of the form $\lim_{x\to 0} f(x)$. In this case, we need to show that $^*f(x) \simeq L$ for all infinitesimal x. We will be able to obtain $^*f(\alpha) =^1 L$ independently of the choice of α, from which is follows that $\lim_{x\to 0} f(x) = L$.

The way in which we will use local equality at first is by using the fact that given $p(x) \in \mathbb{R}[x]$, if we choose $a \in \mathbf{S}' \cup \mathbf{T}'$, then every term of $p(x)$ evaluated at a belongs to a different zone. Thus we have properties such as $p(\omega)$ is locally equal on its own level to the highest degree term of $p(x)$ evaluated at ω.

The following lemma becomes useful in several places where we are working with infinite series of infinitesimals. (The * is omitted from the summations in this section – all summations are nonstandard.)

Lemma 3.1. If β is an infinitesimal, then $\sum_{n=1}^{\infty} \beta^n \mathrel{\widehat{=}} \beta$.

Proof. We have the equation

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\beta^n\right| &\le \sum_{n=1}^{\infty}\left|\beta^n\right| \\ &= |\beta|\sum_{n=1}^{\infty}\left|\beta^{n-1}\right| \\ &< |\beta|\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1} \\ &= 2|\beta| \\ &\mathrel{\hat{=}} \beta. \end{aligned} \tag{3.1}$$

This tells us that $\sum_{n=1}^{\infty}\beta^n \mathrel{\hat{\le}} \beta$. Since β itself is a summand, $\sum_{n=1}^{\infty}\beta^n \mathrel{\hat{=}} \beta$.

In many of the examples in this section, we use power series to represent exponential and trigonometric functions. These series are then evaluated at α, making it possible to use the following lemma.

Lemma 3.2. Suppose $\beta \mathrel{\hat{<}} 1$ and let $\langle a_n \mid n \in \mathbb{N}\rangle$ be a sequence of real numbers such that $|a_n|$ is bounded by some real number m. Then for any finite k, $\sum_{n=1}^{\infty} {}^*a_n\beta^n =^{\beta^k} \sum_{n=1}^{k} {}^*a_n\beta^n$.

Proof. We need to show that $\sum_{n=k+1}^{\infty} {}^*a_n\beta^n \mathrel{\hat{<}} \beta^k$. We can write this sum as

$$\begin{aligned}\left|\sum_{n=k+1}^{\infty} {}^*a_n\beta^n\right| &\le \sum_{n=k+1}^{\infty}\left|{}^*a_n\beta^n\right| \\ &= \left|\beta^k\right|\sum_{n=k+1}^{\infty}\left|{}^*a_n\beta^{n-k}\right| \\ &= \left|\beta^k\right|\sum_{n=1}^{\infty}\left|{}^*a_{n+k}\beta^n\right| \\ &\le m\left|\beta^k\right|\sum_{n=1}^{\infty}\left|\beta^n\right|. \end{aligned} \tag{3.2}$$

By Lemma 3.1, $\sum_{n=1}^{\infty}\left|\beta^n\right| \mathrel{\hat{=}} \beta$. Therefore,

$$m\left|\beta^k\right|\sum_{n=1}^{\infty}\left|\beta^n\right| \mathrel{\hat{=}} \beta^{k+1}. \tag{3.3}$$

This tells us that

$$\left|\sum_{n=k+1}^{\infty} {}^*a_n\beta^n\right| \mathrel{\hat{\leq}} \beta^{k+1}. \tag{3.4}$$

Since $\beta \mathrel{\hat{<}} 1$, $\beta^{k+1} \mathrel{\hat{<}} \beta^k$, so the entire sum is inferior to β^k.

This lemma allows us to make statements such as $\sin\alpha =^{\alpha^3} \alpha - \alpha^3/6$.

3.2 Limits of the Form 0/0 and ∞/∞

For limits which assume the form $0/0$ or ∞/∞, we need to examine the properties of local equality as it pertains to quotients a/b. If $a \mathrel{\hat{<}} b$, then $a/b =^1 0$, and if $a \mathrel{\hat{>}} b$, then a/b is infinite. For the remaining case, $a \mathrel{\hat{=}} b$, we have the following theorem.

Theorem 3.3. Let a_1, a_2, b_1, and b_2 be nonzero elements of ${}^*\mathbb{R}$. Suppose we have the local equalities $a_1 =^{a_1} a_2$ and $b_1 =^{b_1} b_2$. Then $\frac{a_1}{b_1} =^{a_1/b_1} \frac{a_2}{b_2}$.

Proof. Write $a_2 = a_1 + \varepsilon_a$ where $\varepsilon_a \mathrel{\hat{<}} a_1$ and write $b_2 = b_1 + \varepsilon_b$ where $\varepsilon_b \mathrel{\hat{<}} b_1$. Multiplying ε_a by b_1 and ε_b by a_1, we have the relations $b_1\varepsilon_a \mathrel{\hat{<}} a_1b_1$ and $a_1\varepsilon_b \mathrel{\hat{<}} a_1b_1$. Since $W_0(a_1b_1)$ is closed under addition, $a_1\varepsilon_b - b_1\varepsilon_a \mathrel{\hat{<}} a_1b_1$. By Lemma 2.16, $b_1 \mathrel{\hat{=}} b_2$. So we can write

$$a_1\varepsilon_b - b_1\varepsilon_a \mathrel{\hat{<}} a_1b_2. \tag{3.5}$$

The left side of this equation can be rewritten by adding and subtracting a_1b_1 to obtain

$$a_1(b_1+\varepsilon_b) - b_1(a_1+\varepsilon_a) \mathrel{\hat{<}} a_1b_2. \tag{3.6}$$

Replacing $b_1+\varepsilon_b$ with b_2 and $a_1+\varepsilon_a$ with a_2 and then dividing both sides by b_1b_2 (using Lemma 2.7), we have

$$\frac{a_1b_2 - a_2b_1}{b_1b_2} \mathrel{\hat{<}} \frac{a_1}{b_1}. \tag{3.7}$$

The left side of this equation is simply $\frac{a_1}{b_1} - \frac{a_2}{b_2}$, so the proof is complete since this is now the definition of $\frac{a_1}{b_1} =^{a_1/b_1} \frac{a_2}{b_2}$.

A trivial example of the application of Theorem 3.3 is a quotient of polynomials. If we have $\lim_{x\to\infty} p(x)/q(x)$, then we substitute ω for x in both $p(x)$ and

$q(x)$. If $p(\omega) \mathrel{\hat{=}} q(\omega)$, which is true if and only if p and q have the same degree, then the limit is equal to the quotient of the leading coefficients of p and q. Following are two slightly less trivial examples in which trigonometric functions appear.

Example 3.4. $\lim\limits_{x\to\infty} \dfrac{\sin x}{x+x^2}$

Solution. We replace the sine function with its power series and evaluate at α to obtain

$$\frac{\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - + \cdots}{\alpha + \alpha^2}.$$

By Lemma 3.2, $\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - + \cdots =^{\alpha} \alpha$ and $\alpha + \alpha^2 =^{\alpha} \alpha$, so we can apply Theorem 3.3 as follows.

$$\frac{\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - + \cdots}{\alpha + \alpha^2} =^1 \frac{\alpha}{\alpha} = 1.$$

Since this local equality holds for any choice of α, we know $\lim_{x\to 0} \frac{\sin x}{x+x^2} = 1$.

A similar example involving the cosine function is given next.

Example 3.5. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2 - 3x^4}$

Solution. As above, we replace the cosine function with its power series and evaluate at α to obtain

$$\frac{1-\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - + \cdots\right)}{\alpha^2 - 3\alpha^4} = \frac{\frac{\alpha^2}{2!} - \frac{\alpha^4}{4!} + - \cdots}{\alpha^2 - 3\alpha^4}.$$

This time we have $\frac{\alpha^2}{2!} - \frac{\alpha^4}{4!} + - \cdots =^{\alpha^2} \frac{1}{2}\alpha^2$ and $\alpha^2 - 3\alpha^4 =^{\alpha^2} \alpha^2$, so applying Theorem 3.3,

$$\frac{\frac{\alpha^2}{2!} - \frac{\alpha^4}{4!} + - \cdots}{\alpha^2 - 3\alpha^4} =^1 \frac{\frac{1}{2}\alpha^2}{\alpha^2} = \frac{1}{2}.$$

Again, this local equality is independent of α. So $\lim_{x\to 0} \frac{1-\cos x}{x^2 - 3x^4} = \frac{1}{2}$.

Evaluating the limit in Example 3.5 would require two applications of l'Hôpital's rule. However, using local equality enables the limit to be evaluated in a single step once the trigonometric function is replaced by its power series.

3.3 Limits of the Form 1^∞

Far more interesting situations arise when we examine limits which assume the form 1^∞. For this case, we need the following theorem.

Theorem 3.6. Let $a \in {}^*\mathbb{R}$ and suppose $a - 1 \mathrel{\widehat{=}} 1/t$ for some positive infinite number $t \in {}^*\mathbb{R}$. Then for any $b \in {}^*\mathbb{R}$ satisfying $b =^{1/t} a$, we have $a^t =^{a^t} b^t$.

Proof. Let $\varepsilon = b - a$ and assume $\varepsilon \neq 0$. Expanding $(a+\varepsilon)^t$ with the binomial theorem, we obtain

$$b^t = (a+\varepsilon)^t = \sum_{k=0}^{\infty} \binom{t}{k} a^{t-k} \varepsilon^k . \tag{3.8}$$

For $k = 0$, the summand is simply a^t. We wish to show that the sum of the terms for $k \geq 1$ is inferior to a^t. So we rewrite equation (3.8) as

$$b^t = a^t + \sum_{k=1}^{\infty} \binom{t}{k} a^{t-k} \varepsilon^k \tag{3.9}$$

and show that this sum for $k \geq 1$ satisfies the hypothesis of Lemma 3.1. (We will actually be applying this lemma twice.)

The binomial coefficient $\binom{t}{k} = t(t-1)(t-2)\cdots(t-k+1)/k!$ is a degree k polynomial in *t*. When the numerator is expressed as a sum of powers of *t*, the coefficient of the degree *j* term is given by the Stirling number of the first kind $s(k,j)$ where $s(k,j) = 0$ if $j < 1$ or $j > k$. Thus we can write equation (3.9) as

$$b^t = a^t + \sum_{k=1}^{\infty} \left(a^{t-k} \varepsilon^k \sum_{j=1}^{k} \frac{s(k,j)}{k!} t^j \right). \tag{3.10}$$

Since $\varepsilon \mathrel{\widehat{<}} 1/t$, we can write $\varepsilon = \beta/t$ where $\beta \mathrel{\widehat{<}} 1$. Replacing ε with β/t and factoring t^k out of the inner sum, we have

$$b^t = a^t + \sum_{k=1}^{\infty}\left(a^{t-k}\beta^k \sum_{j=1}^{k} \frac{s(k,j)}{k!} t^{j-k} \right). \tag{3.11}$$

We now reverse the order of summation for the inner sum through the map $j \mapsto k+1-j$ so that the terms are arranged in the order necessary for the application of Lemma 3.1. This gives us

$$b^t = a^t + \sum_{k=1}^{\infty}\left(a^{t-k}\beta^k \sum_{j=1}^{k} \frac{s(k,k+1-j)}{k!} t^{1-j} \right). \tag{3.12}$$

For convenience, we define

$$u_k = \sum_{j=1}^{k} \frac{s(k,k+1-j)}{k!} t^{1-j}. \tag{3.13}$$

When k is finite, $u_k \mathrel{\widehat{=}} 1$ since it is a finite sum of elements of $W(1)$. The Stirling numbers of the first kind obey the recurrence relation

$$s(k,j) = s(k-1,j-1) - (k-1)s(k-1,j).$$

Starting with $s(1,1)=1$, an easy induction argument shows that $|s(k,j)| \le k!$ for all k and j. So when k is infinite, we have

$$\begin{aligned} |u_k| &= \sum_{j=1}^{k} \frac{|s(k,k+1-j)|}{k!} t^{1-j} \\ &\le \sum_{j=1}^{k} t^{1-j} \\ &= t\sum_{j=1}^{k} \left(\frac{1}{t}\right)^j \\ &< t\sum_{j=1}^{\infty} \left(\frac{1}{t}\right)^j. \end{aligned} \tag{3.14}$$

Lemma 3.1 applies to the last sum of this equation giving us $\sum_{j=1}^{\infty} t^{-j} \mathrel{\widehat{=}} 1/t$. Therefore, $t\sum_{j=1}^{\infty} t^{-j} \mathrel{\widehat{=}} 1$. Since this expression is greater than $|u_k|$, we must have $u_k \mathrel{\widehat{\le}} 1$. Substituting u_k into equation (3.12) and factoring a^t out of the summation, we have

$$b^t = a^t\left(1+\sum_{k=1}^{\infty}\frac{\beta^k}{a^k}u_k\right). \tag{3.15}$$

Since $u_k \mathrel{\widehat{\leq}} 1$ for all k, we know

$$\begin{aligned}\left|\sum_{k=1}^{\infty}\frac{\beta^k}{a^k}u_k\right| &\leq \sum_{k=1}^{\infty}\left|\frac{\beta^k}{a^k}\right||u_k| \\ &\mathrel{\widehat{\leq}} \sum_{k=1}^{\infty}\left|\frac{\beta^k}{a^k}\right|. \end{aligned}\tag{3.16}$$

Since $\beta \mathrel{\widehat{<}} 1$ and $a \mathrel{\widehat{=}} 1$, we have $\beta/a \mathrel{\widehat{<}} 1$. So Lemma 3.1 applies to this sum and we obtain $\sum_{k=1}^{\infty}\left|\beta^k/a^k\right| \mathrel{\widehat{=}} \beta/a$. Thus, $\sum_{k=1}^{\infty}\left(\beta^k/a^k\right)u_k \mathrel{\widehat{\leq}} \beta/a \mathrel{\widehat{<}} 1$. Calling this sum ξ, we now have $b^t = a^t(1+\xi) = a^t + a^t\xi$ where $a^t\xi \mathrel{\widehat{<}} a^t$. So $a^t =^{a^t} b^t$ and the theorem is proven.

The simplest application of Theorem 3.6 is to the evaluation of $\lim_{x\to\infty}(1+c/x)^x$. When we substitute ω for x, we have $(1+c\alpha)^\omega$. Since $e^{c\alpha} = 1+c\alpha+c^2\alpha^2/2!+-\cdots$, we have $1+c\alpha =^{\alpha} e^{c\alpha}$. Therefore, by Theorem 3.6 and since $e^{c\alpha\omega} = e^c$, we have $(1+c/\alpha)^\omega =^{e^c} e^c$, which is equivalent to $(1+c/\alpha)^\omega =^1 e^c$. This happens independently of our choice of ω, so $\lim_{x\to\infty}(1+c/x)^x = e^c$.

Theorem 3.6 tells us much more than this. In fact, it is now easy to show that for any $f(x)$ satisfying $*f(\omega) \mathrel{\widehat{<}} \alpha$, the limit $\lim_{x\to\infty}(1+c/x+f(x))^x$ does not differ from $\lim_{x\to\infty}(1+c/x)^x$.

We can also use Theorem 3.6 to more easily evaluate limits of this form where c is a function of x instead of a constant. This is demonstrated in the first example below.

Example 3.7. $\lim\limits_{x\to\infty}\dfrac{\left(1+\frac{\ln x}{x}\right)^x}{x+1}$

Solution. We first evaluate at ω to obtain the expression

$$\frac{(1+\alpha\ln\omega)^\omega}{\omega+1}.$$

Since $\ln\omega \in \text{zone}(1)$, local equality on the level of α is equivalent to local equality on the level of $\alpha\ln\omega$. So $1+\alpha\ln\omega =^{\alpha} e^{\alpha\ln\omega} = \omega^\alpha$, from which Theorem 3.6 tells us that $(1+\alpha\ln\omega)^{\omega} =^{\omega} \omega$. We also have for the denominator $\omega+1 =^{\omega} \omega$, so by Theorem 3.3,

$$\frac{(1+\alpha\ln\omega)^{\omega}}{\omega+1} =^{1} 1.$$

Therefore, the value of the limit is 1.

Example 3.8. $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$

Solution. Evaluating at α, we have

$$\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)^{\omega}.$$

Using the identity $c^{\alpha} = e^{\alpha\ln c}$, this can be rewritten as

$$\left(\frac{e^{\alpha\ln a}+e^{\alpha\ln b}}{2}\right)^{\omega}.$$

Adding the power series for the exponentials together, we obtain

$$\left(1+\frac{1}{2}\alpha(\ln a+\ln b)+\frac{1}{4}\alpha^2\left((\ln a)^2+(\ln b)^2\right)+\cdots\right)^{\omega}.$$

Even though the above expression is not the power series for $e^{(\ln a+\ln b)/2} = \sqrt{ab}$, it is locally equal to this power series on the level of α. So we can apply Theorem 3.6 to obtain

$$\left(1+\frac{1}{2}\alpha(\ln a+\ln b)+\frac{1}{4}\alpha^2\left((\ln a)^2+(\ln b)^2\right)+\cdots\right)^{\omega} =^{\sqrt{ab}} \sqrt{ab}.$$

Therefore, the value of the limit is $\sqrt{ab}$.

Example 3.9. $\lim_{x\to 0}(\cos x)^{\cot^2 x}$

Solution. Evaluating at α and substituting power series, we have

$$(\cos\alpha)^{\cot^2\alpha} = \left(1-\frac{\alpha^2}{2!}+-\cdots\right)^{\left(\frac{1-\alpha^2/2!+-\cdots}{\alpha-\alpha^3/3!+-\cdots}\right)^2}.$$

For the exponent we can use Theorem 3.3 to obtain $\frac{1-\alpha^2/2!+-\cdots}{\alpha-\alpha^3/3!+-\cdots} =^1 \omega$ since $\cos\alpha =^1 1$ and $\sin\alpha =^1 \alpha$. So we write $t = \cot^2\alpha = \omega^2 + \beta$ where $\beta \mathrel{\widehat{<}} \omega$. Now for the base we have $\cos\alpha - 1 \mathrel{\widehat{=}} \alpha^2 \mathrel{\widehat{=}} 1/t$. All of this shows that $(\cos\alpha)^{\cot^2\alpha}$ satisfies the conditions for Theorem 3.6. Since $e^{-\alpha^2/2} =^{\alpha^2} \cos\alpha$ and $\cos\alpha \mathrel{\widehat{=}} 1$, Theorem 3.6 tells us that

$$\begin{aligned}(\cos\alpha)^{\cot^2\alpha} &=^1 \left(e^{-\alpha^2/2}\right)^{\cot^2\alpha}\\ &= \left(e^{-\alpha^2/2}\right)^{\omega^2+\beta}\\ &= e^{-1/2}e^{-\alpha^2\beta/2}.\end{aligned}$$

Since $\beta \mathrel{\widehat{<}} \omega$, $-\alpha^2\beta/2$ is an infinitesimal which we will call γ. We now have

$$\begin{aligned}e^{-1/2}e^{-\alpha^2\beta/2} &= e^{-1/2}e^{\gamma}\\ &= e^{-1/2}\left(1+\sum_{n=1}^{\infty}\frac{\gamma^n}{n!}\right).\end{aligned}$$

Using Lemma 3.1,

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\frac{\gamma^n}{n!}\right| &\le \sum_{n=1}^{\infty}\left|\frac{\gamma^n}{n!}\right|\\ &< \sum_{n=1}^{\infty}\left|\gamma^n\right|\\ &\mathrel{\widehat{=}} \gamma.\end{aligned}$$

So we finally have

$$e^{-1/2}\left(1+\sum_{n=1}^{\infty}\frac{\gamma^n}{n!}\right) =^1 e^{-1/2}.$$

Therefore, the value of the limit is $e^{-1/2}$.

These examples would require a great deal more work if we were to use l'Hôpital's rule to evaluate the limits. However, to one who is adept at using Theorems 3.3 and 3.6, these limits can be evaluated with far less effort.

3.4 Uncompensated Square Completion

Another interesting situation arises for some limits which assume the form $\infty - \infty$. The example following the next theorem shows that we may sometimes add a number to an expression in order to complete a square without ever subtracting the number elsewhere—that is, we never have to compensate for the change to the expression.

Theorem 3.10. Let t be a positive infinite number and let $c \mathrel{\widehat{<}} \sqrt{t}$. Then $\sqrt{t+c} =^1 \sqrt{t}$.

Proof. Expanding $\sqrt{t+c}$ with the binomial theorem we have

$$\begin{aligned}\sqrt{t+c} &= \sum_{k=0}^{\infty}\binom{1/2}{k} t^{1/2-k}c^k \\ &= \sqrt{t} + \sqrt{t}\sum_{k=1}^{\infty}\binom{1/2}{k}\frac{c^k}{t^k}. \qquad (3.17)\end{aligned}$$

Since $\left|\binom{1/2}{k}\right| < 1$ for all k, we can write

$$\left|\sum_{k=1}^{\infty}\binom{1/2}{k}\frac{c^k}{t^k}\right| \le \sum_{k=1}^{\infty}\left|\frac{c^k}{t^k}\right|. \qquad (3.18)$$

Since $c \mathrel{\widehat{<}} t$, c/t is an infinitesimal. So Lemma 3.1 applies to this sum giving us $\sum_{k=1}^{\infty}\left|\left(c^k/t^k\right)\right| \mathrel{\widehat{=}} c/t$. We now have

$$\sqrt{t}\left|\sum_{k=1}^{\infty}\binom{1/2}{k}\frac{c^k}{t^k}\right| \mathrel{\widehat{\le}} \frac{c}{\sqrt{t}}. \qquad (3.19)$$

Since $c \mathrel{\widehat{<}} \sqrt{t}$, $c/\sqrt{t} \mathrel{\widehat{<}} 1$. Therefore, from equation (3.17), $\sqrt{t+c} = \sqrt{t} + \xi$ where $\xi \mathrel{\widehat{<}} 1$. So $\sqrt{t+c} =^1 \sqrt{t}$.

Example 3.11. $\lim_{x\to\infty}\sqrt{x^2+6x} - x$

Solution. Evaluating at ω, we have $\sqrt{\omega^2 + 6\omega} - \omega$. Theorem 3.10 tells us that for any $c \mathrel{\overset{\frown}{<}} \sqrt{\omega^2 + 6\omega}$, we must have $\sqrt{\omega^2 + 6\omega + c} =^1 \sqrt{\omega^2 + 6\omega}$. So we choose the only value of c that is of any advantage—the one which completes the square under the radical. Setting $c = 9$, we have

$$\begin{aligned} \sqrt{\omega^2 + 6\omega} - \omega &=^1 \sqrt{\omega^2 + 6\omega + 9} - \omega \\ &= \sqrt{(\omega + 3)^2} - \omega \\ &= 3. \end{aligned}$$

Therefore, the value of the limit is 3.

This method of uncompensated square completion provides a much faster alternative to the standard method of multiplying the expression by its conjugate.

4 Infinite Series

In this section we present two new tests for the convergence of infinite series which are analogs of the comparison test and the limit comparison test of standard analysis. Once these tests have been introduced, we examine some useful facts which allow easier application of the tests and present some examples.

4.1 Convergence Tests

Both of the new tests determine the convergence of a series $\sum_{n=1}^{\infty} f(n)$ by examining the properties of $*f(\omega)$. The first test checks to see whether $*g(\omega) \hat{<} *f(\omega)$ for some convergent series $\sum_{n=1}^{\infty} f(n)$.

> **Theorem 4.1 (Order Comparison Test).** Let $f(x)$ and $g(x)$ be standard positive-valued functions. If the series $\sum_{n=1}^{\infty} f(n)$ converges and $*g(\omega) \hat{<} *f(\omega)$ (independent on the choice of ω) then the series $\sum_{n=1}^{\infty} g(n)$ also converges.
>
> **Proof.** Since $*g(\omega) \hat{<} *f(\omega)$, we know that $*g(\omega)/*f(\omega) \hat{<} 1$ and thus $*g(\omega)/*f(\omega)$ is an infinitesimal. Since this happens independently of the choice of ω, this implies that $\lim_{x\to\infty} g(x)/f(x) = 0$. Therefore, given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $g(n)/f(n) < \varepsilon$. Thus for all $n > N$, $g(n) < \varepsilon f(n)$. Since $\sum_{n=1}^{\infty} \varepsilon f(n)$ converges, $\sum_{n=1}^{\infty} g(n)$ also converges by the comparison test.

Note that the contrapositive of this theorem states that if the series $\sum_{n=1}^{\infty} g(n)$ diverges and $*g(\omega) \hat{<} *f(\omega)$, then the series $\sum_{n=1}^{\infty} f(n)$ also diverges.

Let $z = \operatorname{ord}(*g(\omega))$ for some standard positive-valued function g. It immediately follows from the order comparison test that if $z \underset{\not\sim}{<} -1$ then the series $\sum_{n=1}^{\infty} g(n)$ converges since in this case we would have $*g(\omega) \hat{<} 1/\omega^p$ for some real number p with $1 < p < -z$. If $z \underset{\not\sim}{>} -1$, then the series $\sum_{n=1}^{\infty} g(n)$ diverges since in this case we would have $*g(\omega) \hat{>} 1/\omega^p$ for some real p with $-z < p < 1$.

For the remaining case, $z \simeq 1$, we cannot conclude anything from the order comparison test. This means that if $*f(\omega) \hat{=} *g(\omega)$, then whether $\sum_{n=1}^{\infty} f(n)$

converges tells us nothing about whether $\sum_{n=1}^{\infty} g(n)$ converges. However, if $*f(\omega) =^{*f(\omega)} *g(\omega)$ then we can infer the convergence of one series from the other. For this situation, we have the following test.

Theorem 4.2 (Local Equality Test). If f and g are standard positive-valued functions that satisfy $*f(\omega) =^{*f(\omega)} *g(\omega)$ (independent on the choice of ω), then the series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} g(n)$ either both converge or both diverge.

Proof. Since $*f(\omega) =^{*f(\omega)} *g(\omega)$, we know that $*g(\omega)/*f(\omega) =^{1} 1$, which implies that $*g(\omega)/*f(\omega) \simeq 1$. Since this property is independent of the choice of ω, this means that $\lim_{x\to\infty} g(x)/f(x) = 1$. Therefore, by the limit comparison test, the series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} g(n)$ either both converge or both diverge.

4.2 Using the New Tests

It is usually more convenient to think of an infinite series as a sum of the reciprocals of a function evaluated at each natural number. Fortunately, the order comparison test and local equality test work equally well for this situation. This is because $*f(\omega) \hat{>} *g(\omega)$ implies $1/*f(\omega) \hat{<} 1/*g(\omega)$ and $*f(\omega) =^{*f(\omega)} *g(\omega)$ implies $1/*f(\omega) =^{1/*f(\omega)} 1/*g(\omega)$.

In many cases when the order comparison test cannot be used, it will be possible through local equality to reduce the number $*f(\omega)$ to a number of the form $1/a\omega$ where $a \in \text{zone}(1)$. When this happens, the following extension to the p-series test is useful. We use the notation $\ln^{(n)}$ to mean the natural logarithm taken n times (e.g., $\ln^{(3)} x = \ln\ln\ln x$).

Let $k \in \mathbb{N}$ and let m be the least natural number for which $\ln^{(k)} m$ is real and greater than 1. Then the series

$$\sum_{n=m}^{\infty} \frac{1}{n \ln n \ln^{(2)} n \cdots \left(\ln^{(k)} n\right)^{p}}$$

converges if and only if $p > 1$. We show this by using the integral test. The integral

$$\int_m^\infty \frac{1}{x \ln x \ln^{(2)} x \cdots \left(\ln^{(k)} x\right)^p} dx$$

can be evaluated by making the substitution $u = \ln^{(k)} x$ for which we have $du = 1/x \ln x \ln^{(2)} x \cdots \ln^{(k-1)} x$. This gives us the integral

$$\int_{\ln^{(k)} m}^\infty \frac{du}{u^p},$$

which converges if and only if $p > 1$.

We conclude this section with a few examples.

Example 4.3. $\displaystyle\sum_{n=1}^\infty \frac{n^2 - 3n + 2}{4n^3 + 6n^2 - 1}$

Solution. Let $f(n) = \frac{4n^3+6n^2-1}{n^2-3n+2}$. We substitute ω for n and notice

$$\frac{4\omega^3 + 6\omega^2 - 1}{\omega^2 - 3\omega + 2} \widehat{=} \omega.$$

So if we can find a function $g(n)$ such that $1/{*g(\omega)} =^\omega \frac{4\omega^3+6\omega^2-1}{\omega^2-3\omega+2}$ for which we know whether $\sum_{n=1}^\infty 1/g(n)$ converges, then we can use the local equality test to determine whether $\sum_{n=1}^\infty 1/f(n)$ converges. By Theorem 3.3,

$$\frac{4\omega^3 + 6\omega^2 - 1}{\omega^2 - 3\omega + 2} =^\omega 4\omega$$

since $4\omega^3 + 6\omega^2 - 1 =^{\omega^3} 4\omega^3$ and $\omega^2 - 3\omega + 2 =^{\omega^2} \omega^2$. Because $\sum_{n=1}^\infty 1/4n$ diverges, the local equality test tells us that the series that we are testing also diverges.

The next example demonstrates the procedure for dealing with factorials. As shown, the Stirling approximation for the factorial gives a good representation of the size of $\omega!$.

Example 4.4. $\displaystyle\sum_{n=1}^\infty \frac{n^5}{n!}$

Solution. From Stirling's approximation to $n!$, we know that

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}\, n^n e^{-n}}{n!} = 1.$$

Therefore $\omega! \simeq \sqrt{2\pi\omega}\,\omega^\omega e^{-\omega}$. Using this, we can write

$$\frac{\omega^5}{\omega!} \mathrel{\widehat{=}} \frac{\omega^5}{\omega^{\omega+\frac{1}{2}} e^{-\omega}}$$
$$= \frac{1}{\omega^{\omega\left(1-\frac{9}{2}\alpha-\frac{1}{\ln\omega}\right)}}.$$

Since $1-\frac{9}{2}\alpha-\frac{1}{\ln\omega} \in \text{zone}(1)$, the exponent of ω in this last expression is still an infinite number, so $\omega^{\omega\left(1-\frac{9}{2}\alpha-\frac{1}{\ln\omega}\right)} \mathrel{\widehat{>}} \omega^2$. Therefore, by the order comparison test, the series converges.

The above example may seem like a lot of work for such a simple summand. The intent was to demonstrate that $\omega! \mathrel{\widehat{>}} \omega^p$ for *any* finite number p. This fact will usually be enough to tell quickly whether a series containing factorials converges.

We finish with a short example.

Example 4.5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2 - \ln n}$

Solution. The local equality test tells us that this series converges if and only if the series $\sum_{n=1}^{\infty} 1/n(\ln n)^2$ converges, which it does by the earlier remark pertaining to the extended p-series test.

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List of Notation

$\langle x_n \rangle \equiv \langle y_n \rangle$	$x_n = y_n$ almost everywhere
$[\langle a_n \rangle]$	$\{\langle x_n \rangle \mid \langle x_n \rangle \equiv \langle a_n \rangle\}$
$*A$	*-transform of A; $\{[\langle a_n \rangle] \mid a_n \in A$ almost everywhere$\}$
$*\mathbb{N}$	Hypernatural numbers
$*\mathbb{Z}$	Hyperintegers
$*\mathbb{Q}$	Hyperrational numbers
$*\mathbb{R}$	Hyperreal numbers
$\mathbf{T}$	Set of all infinite numbers
$\mathbf{S}$	Set of all infinitesimals
A_∞	Set of infinite numbers in A; $A \cap \mathbf{T}$
$x \simeq y$	x is infinitely close to y; $x - y \in \mathbf{S}$
$x \lesssim y$	$x < y$ or $x \simeq y$
$x \gtrsim y$	$x > y$ or $x \simeq y$
$x \lnsim y$	$x < y$ and $x \not\simeq y$
$x \gnsim y$	$x > y$ and $x \not\simeq y$
$m(a)$	Monad about a; $\{x \mid x \simeq a\}$
$G(a)$	Galaxy about a; $\{x \mid x - a$ is finite$\}$
$\mathrm{ord}(a)$	Order of a; $\log_\omega \lvert a \rvert$
$\mathbf{T}'$	$\{x \in \mathbf{T} \mid \mathrm{ord}(x) \notin \mathbf{S}\}$
$\mathbf{S}'$	$\{x \in \mathbf{S} \mid \mathrm{ord}(x) \notin \mathbf{S}\}$
$x \,\hat{=}\, y$	x is isometric to y; $\mathrm{ord}(x) \simeq \mathrm{ord}(y)$
$x \,\hat{<}\, y$	x is inferior to y; $\mathrm{ord}(x) \lnsim \mathrm{ord}(y)$
$x \,\hat{>}\, y$	x is superior to y; $\mathrm{ord}(x) \gnsim \mathrm{ord}(y)$
$x \,\hat{\leq}\, y$	$x \,\hat{<}\, y$ or $x \,\hat{=}\, y$
$x \,\hat{\geq}\, y$	$x \,\hat{>}\, y$ or $x \,\hat{=}\, y$
$\mathrm{zone}(a)$	Zone about a; $\{x \mid x \,\hat{=}\, a\}$
$W(a)$	World about a; $\{x \mid x \,\hat{\leq}\, a\}$
$W_0(a)$	$\{x \mid x \,\hat{<}\, a\}$
$x =^{\ell} y$	x is locally equal to y on the level of ℓ; $x - y \,\hat{<}\, \ell$