Projective Geometric Algebra and Relativistic Quaternions

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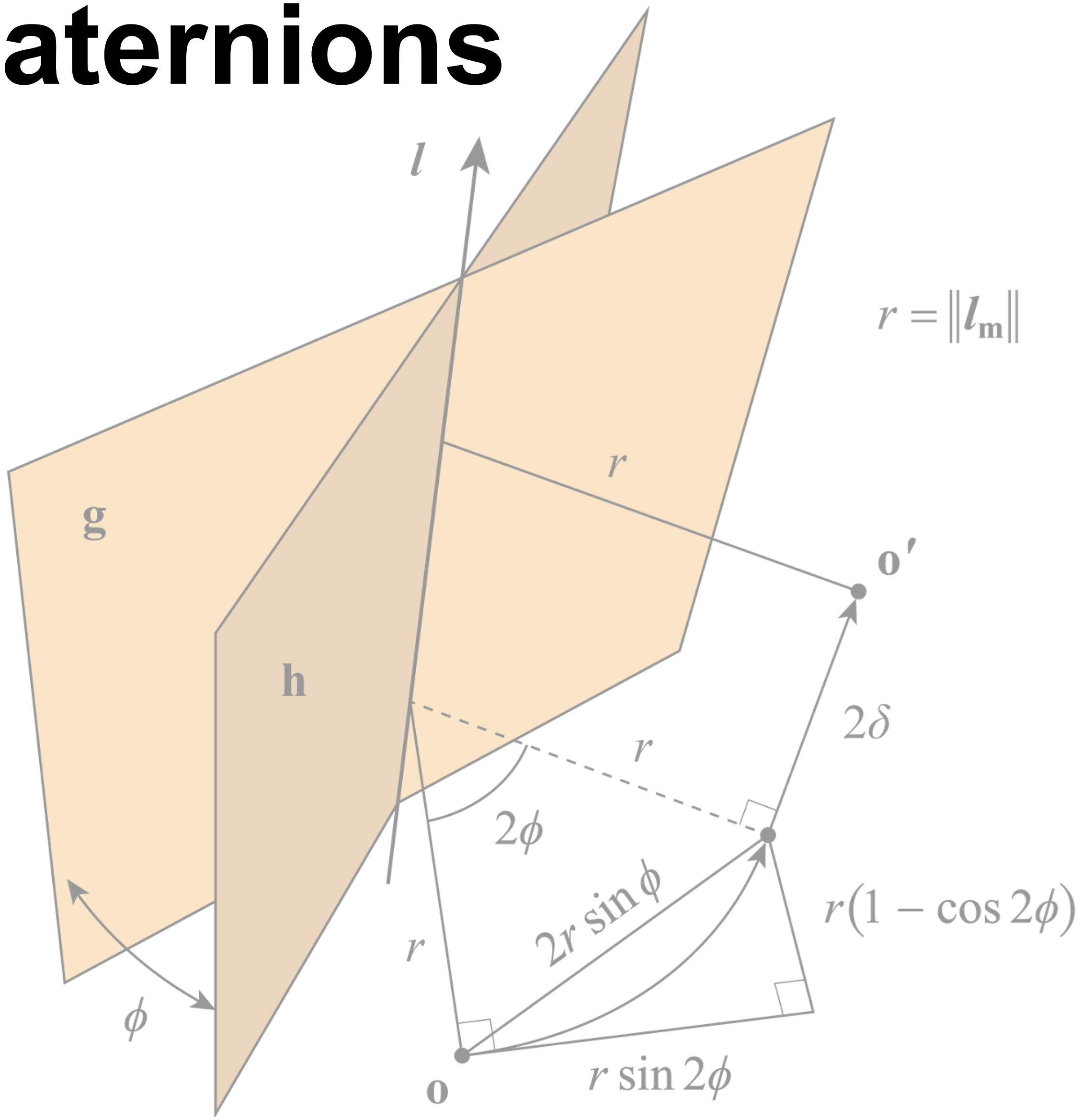
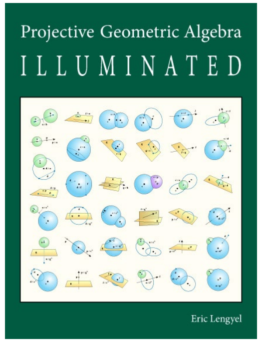
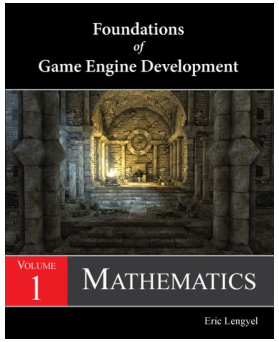
Equations by Radical Pie

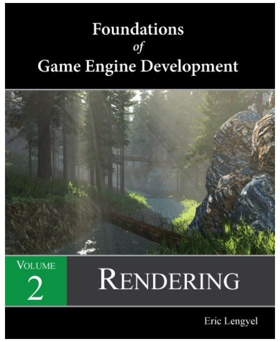


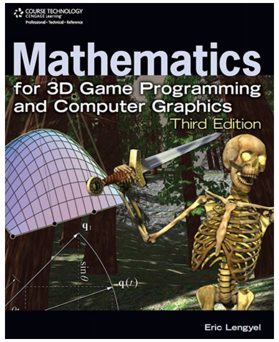
About the Speaker

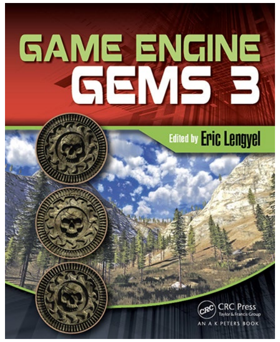
- Virginia Tech mathematics alumnus, B.S. 1994, M.S. 1996
- UC Davis computer science, Ph.D. 2010
- Working in industry since 1994 (former Sierra, Apple, Sony)
- Developing algebraic models for about 15 years
- Occasionally teaches computer graphics
- Writes books about math and real-time rendering
- Mathematical typography expert











Equations by Radical Pie

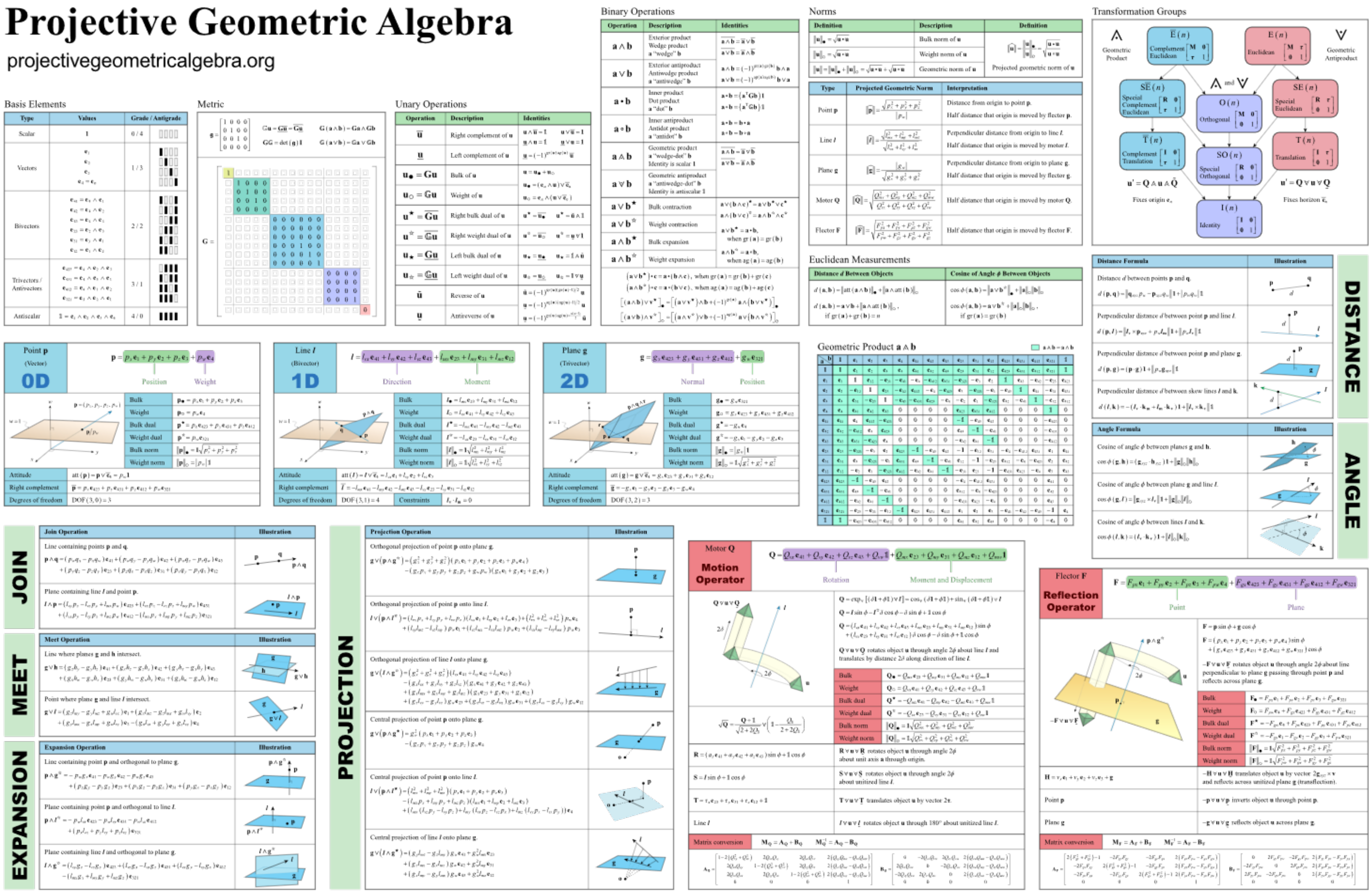
A Vast Subject Area

- No hope of covering all the fundamentals in one hour

- This talk is an introduction that paints the big picture

Projective Geometric Algebra

projectivegeometricalgebra.org



Basis Elements

Type	Values	Grade / Antigrade	
Scalar	1	0 / 4	▯▯▯▯
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4 = \mathbf{e}_w$	1 / 3	▮▯▯▯ ▯▮▯▯ ▯▯▮▯ ▯▯▯▮
Bivectors	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$	2 / 2	▮▯▯▮ ▯▮▯▮ ▯▯▮▮ ▯▮▮▯ ▮▯▮▯ ▮▮▯▯
Trivectors / Antivectors	$\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$	3 / 1	▯▮▮▮ ▮▯▮▮ ▮▮▯▮ ▮▮▮▯
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	▮▮▮▮

Metric

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$G\mathbf{u} = \overline{G\underline{\mathbf{u}}} = \underline{G\overline{\mathbf{u}}}$

$GG = \det(\mathfrak{g})\,\mathbf{I}$

$G(\mathbf{a} \wedge \mathbf{b}) = G\mathbf{a} \wedge G\mathbf{b}$

$G(\mathbf{a} \vee \mathbf{b}) = G\mathbf{a} \vee G\mathbf{b}$

$G =$

Unary Operations

Binary Operations

Norms

Transformation Groups

Euclidean Measurements

Point **p** (Vector) **0D**

$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$

Position Weight

Line *l* (Bivector) **1D**

Direction Moment

Plane **g** (Trivector) **2D**

Normal Position

Geometric Product $\mathbf{a} \mathbin{⟑} \mathbf{b}$

DISTANCE

ANGLE

JOIN

MEET

EXPANSION

PROJECTION

Motor **Q**

Motion Operator

Rotation Moment and Displacement

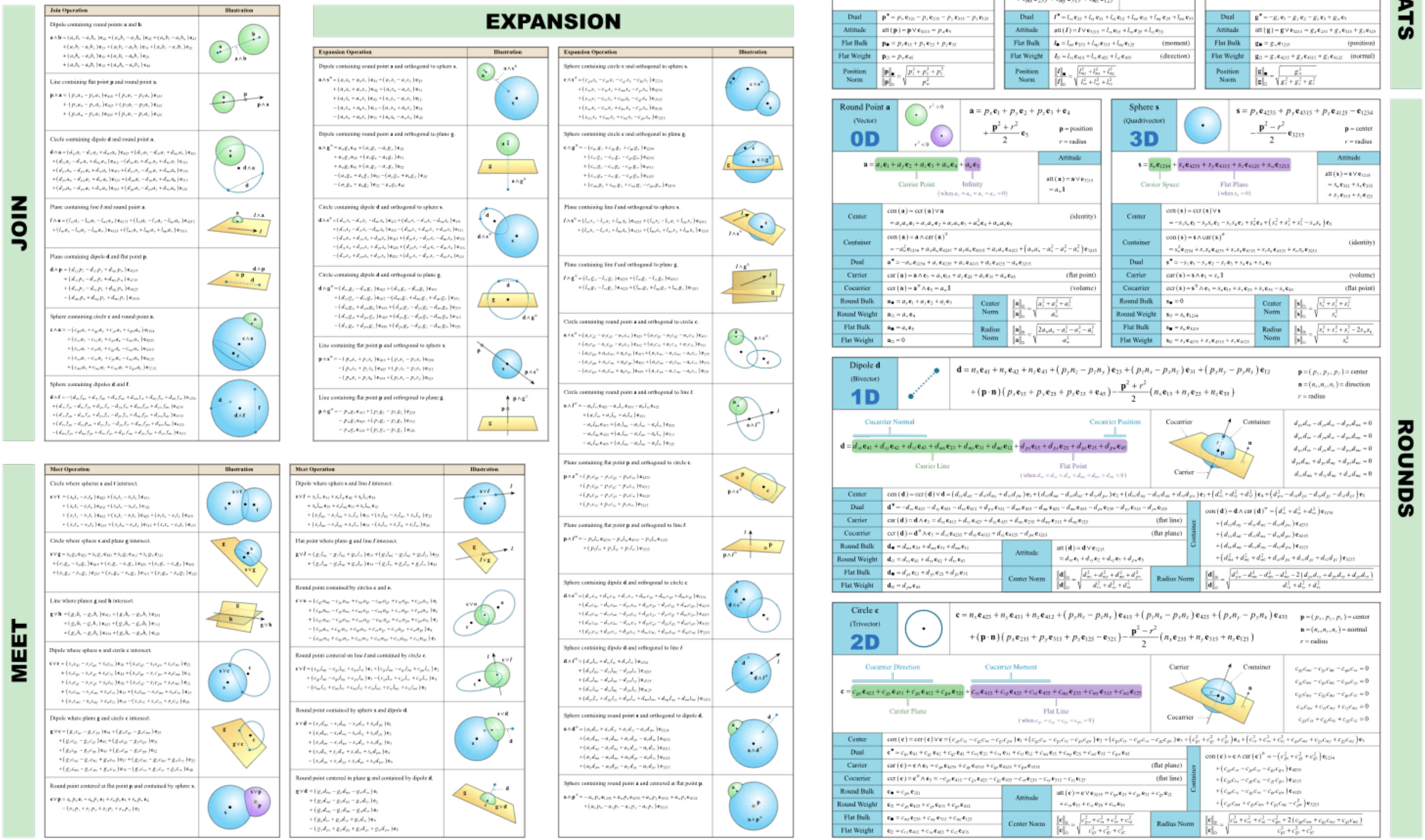
Flector **F**

Reflection Operator

Point Plane

Conformal Geometric Algebra

conformalgeometricalgebra.org



Grassmann / Clifford Algebras

- You’ve probably been using pieces of these algebras already without realizing it

- Cross products
- Homogeneous coordinates (*x*, *y*, *z*, *w*)
- Planes (*a*, *b*, *c*, *d*)
- Plücker coordinates
- Quaternions

Cross Products

- Units of distance become units of area

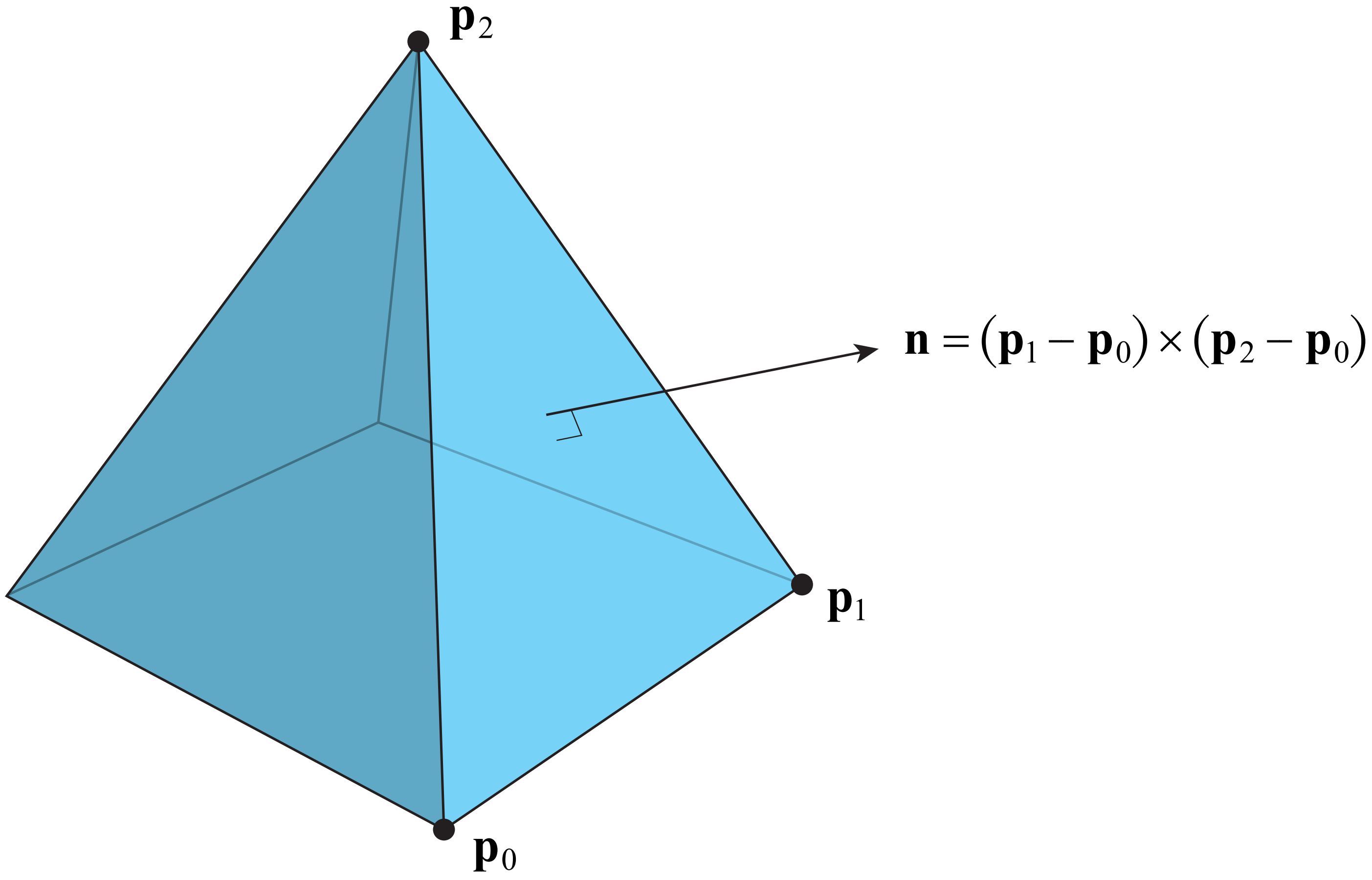
$$(a_x, a_y, a_z) \times (b_x, b_y, b_z)$$

$$\Big\downarrow$$

$$(a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x)$$

Normal Vectors

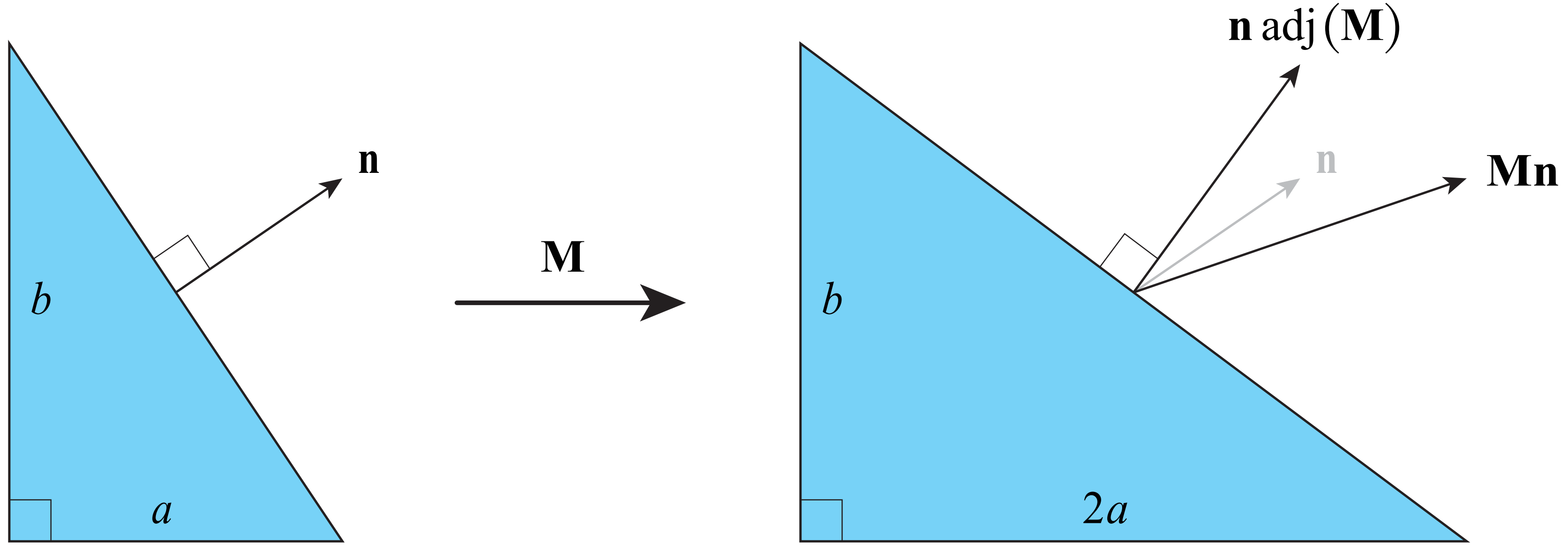
- Cross product calculates normal of triangular face



Normal Vector Transformation

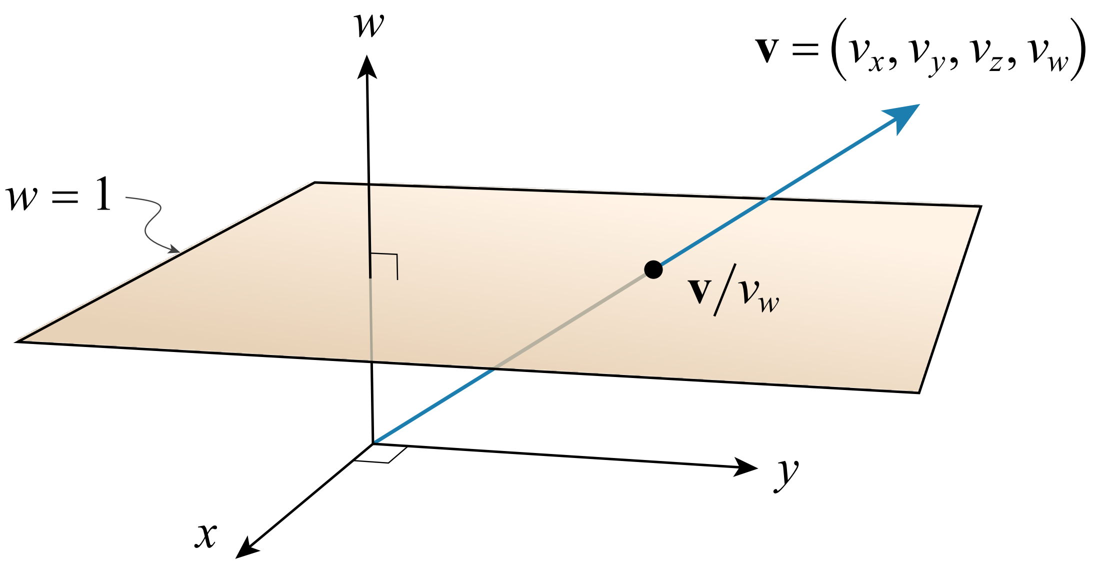
- Normals don’t transform like ordinary vectors
- That’s because they’re something else called *bivectors*

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Homogeneous Coordinates

- 3D points are projections of 4D vectors



Homogeneous Coordinates

- Allows translations to be added to linear transformations

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

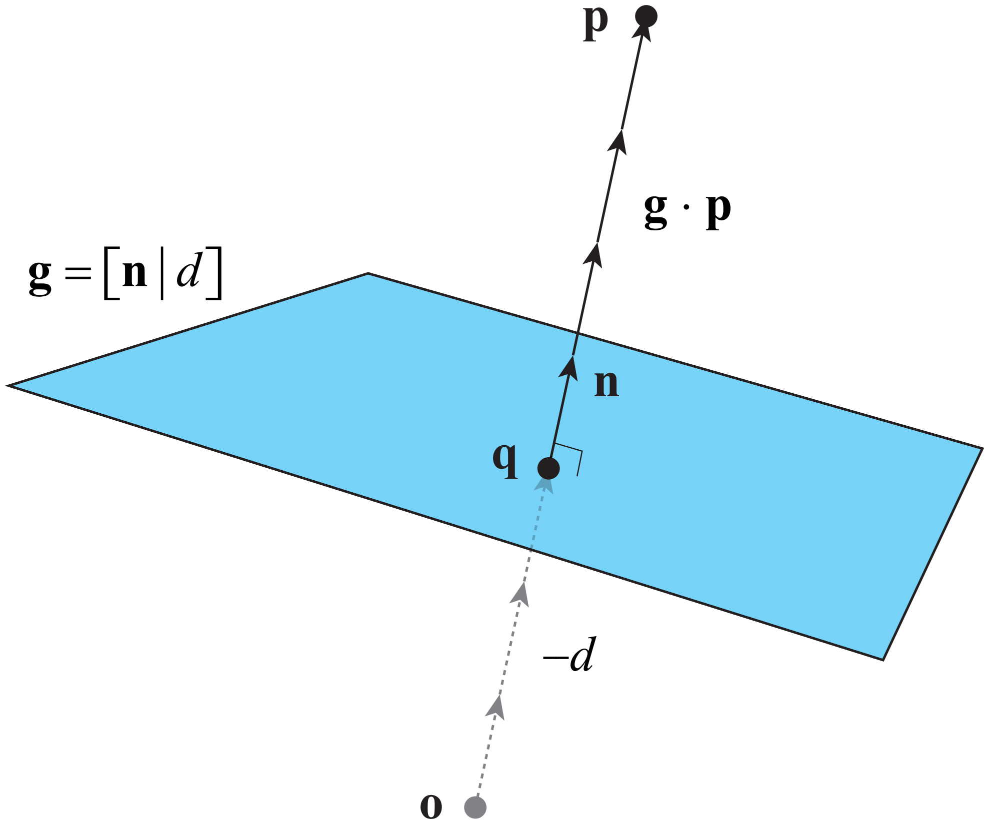
$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & t_x \\ m_{21} & m_{22} & m_{23} & t_y \\ m_{31} & m_{32} & m_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Planes

- 4D dot product with point **p** gives signed distance to plane **g**

$$\mathbf{p} = (x, y, z, w)$$

$$\mathbf{g} = (n_x, n_y, n_z, d)$$



Plücker Coordinates

- Implicit representation of a line in 3D space
- Has 6 coordinates, 3 for direction **v** and 3 for moment **m**

- Given homogeneous points **p** and **q** on the line,

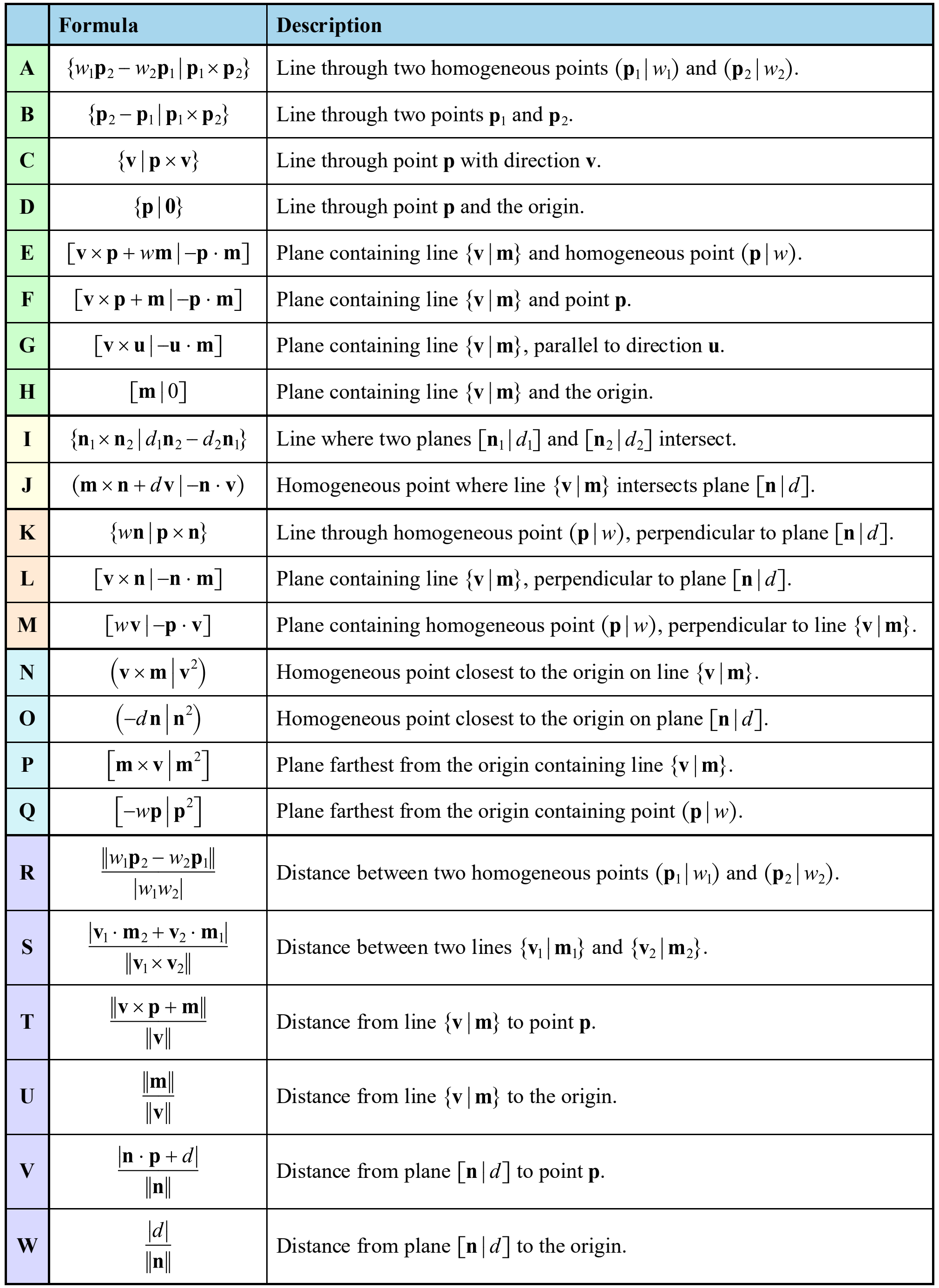
$$\mathbf{v} = p_w \mathbf{q}_{xyz} - q_w \mathbf{p}_{xyz}$$

$$\mathbf{m} = \mathbf{p}_{xyz} \times \mathbf{q}_{xyz}$$

- Same results for any two points spaced same distance apart
- Information about specific points is eliminated

Points, Lines, Planes

- Lots of formulas for combining geometries
- Discovered without knowledge of bigger picture
- We can better explain where all of these formulas come from



	Formula	Description
A	$\{w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two homogeneous points $(\mathbf{p}_1 \mid w_1)$ and $(\mathbf{p}_2 \mid w_2)$.
B	$\{\mathbf{p}_2 - \mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two points $\mathbf{p}_1$ and $\mathbf{p}_2$.
C	$\{\mathbf{v} \mid \mathbf{p} \times \mathbf{v}\}$	Line through point $\mathbf{p}$ with direction $\mathbf{v}$.
D	$\{\mathbf{p} \mid \mathbf{0}\}$	Line through point $\mathbf{p}$ and the origin.
E	$[\mathbf{v} \times \mathbf{p} + w\mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and homogeneous point $(\mathbf{p} \mid w)$.
F	$[\mathbf{v} \times \mathbf{p} + \mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and point $\mathbf{p}$.
G	$[\mathbf{v} \times \mathbf{u} \mid -\mathbf{u} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$, parallel to direction $\mathbf{u}$.
H	$[\mathbf{m} \mid 0]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and the origin.
I	$\{\mathbf{n}_1 \times \mathbf{n}_2 \mid d_1\mathbf{n}_2 - d_2\mathbf{n}_1\}$	Line where two planes $[\mathbf{n}_1 \mid d_1]$ and $[\mathbf{n}_2 \mid d_2]$ intersect.
J	$(\mathbf{m} \times \mathbf{n} + d\mathbf{v} \mid -\mathbf{n} \cdot \mathbf{v})$	Homogeneous point where line $\{\mathbf{v} \mid \mathbf{m}\}$ intersects plane $[\mathbf{n} \mid d]$.
K	$\{w\mathbf{n} \mid \mathbf{p} \times \mathbf{n}\}$	Line through homogeneous point $(\mathbf{p} \mid w)$, perpendicular to plane $[\mathbf{n} \mid d]$.
L	$[\mathbf{v} \times \mathbf{n} \mid -\mathbf{n} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$, perpendicular to plane $[\mathbf{n} \mid d]$.
M	$[w\mathbf{v} \mid -\mathbf{p} \cdot \mathbf{v}]$	Plane containing homogeneous point $(\mathbf{p} \mid w)$, perpendicular to line $\{\mathbf{v} \mid \mathbf{m}\}$.
N	$(\mathbf{v} \times \mathbf{m} \mid \mathbf{v}^2)$	Homogeneous point closest to the origin on line $\{\mathbf{v} \mid \mathbf{m}\}$.
O	$(-d\mathbf{n} \mid \mathbf{n}^2)$	Homogeneous point closest to the origin on plane $[\mathbf{n} \mid d]$.
P	$[\mathbf{m} \times \mathbf{v} \mid \mathbf{m}^2]$	Plane farthest from the origin containing line $\{\mathbf{v} \mid \mathbf{m}\}$.
Q	$[-w\mathbf{p} \mid \mathbf{p}^2]$	Plane farthest from the origin containing point $(\mathbf{p} \mid w)$.
R	$\dfrac{\lVert w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \rVert}{\lvert w_1 w_2 \rvert}$	Distance between two homogeneous points $(\mathbf{p}_1 \mid w_1)$ and $(\mathbf{p}_2 \mid w_2)$.
S	$\dfrac{\lvert \mathbf{v}_1 \cdot \mathbf{m}_2 + \mathbf{v}_2 \cdot \mathbf{m}_1 \rvert}{\lVert \mathbf{v}_1 \times \mathbf{v}_2 \rVert}$	Distance between two lines $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ and $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$.
T	$\dfrac{\lVert \mathbf{v} \times \mathbf{p} + \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to point $\mathbf{p}$.
U	$\dfrac{\lVert \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to the origin.
V	$\dfrac{\lvert \mathbf{n} \cdot \mathbf{p} + d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to point $\mathbf{p}$.
W	$\dfrac{\lvert d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to the origin.

Quaternions

- A quaternion **q** represents a rotation in 3D space

$$\mathbf{q} = xi + yj + zk + w \qquad i^2 = j^2 = k^2 = -1$$

$$\begin{aligned} ij &= -ji = k \\ jk &= -kj = i \\ ki &= -ik = j \end{aligned}$$

- Rotation through angle ϕ about axis **a** is

$$\mathbf{q} = \left(\sin\frac{\phi}{2}\right)\mathbf{a} + \cos\frac{\phi}{2}$$

Quaternions

- A quaternion rotates a vector **v** with the sandwich product

$$\mathbf{v}' = \mathbf{q}\mathbf{v}\mathbf{q}^* \qquad \mathbf{v} = v_x i + v_y j + v_z k$$

- **q*** is the conjugate of the quaternion:

$$\mathbf{q}^* = -xi - yj - zk + w$$

All Part of Same Algebraic Structure

- Non-vector result of cross product
- 4D homogeneous coordinates for points
- 6D Plücker coordinates for lines
- 4D plane representations
- Quaternions

4D Associative Projective Algebras

- 4D rigid exterior algebra
  - Homogeneous representation of 3D geometry
  - Points, lines, planes
  - Join, meet, projection, norm, distance, angle

- 4D rigid geometric algebra
  - Euclidean isometries in 3D space
  - Rotations, translations, screw transformations
  - Parameterization, interpolation

Exterior / Grassmann Algebra

- Wedge product $\wedge$
  - Combines dimensions of operands
  - Vectors square to zero:

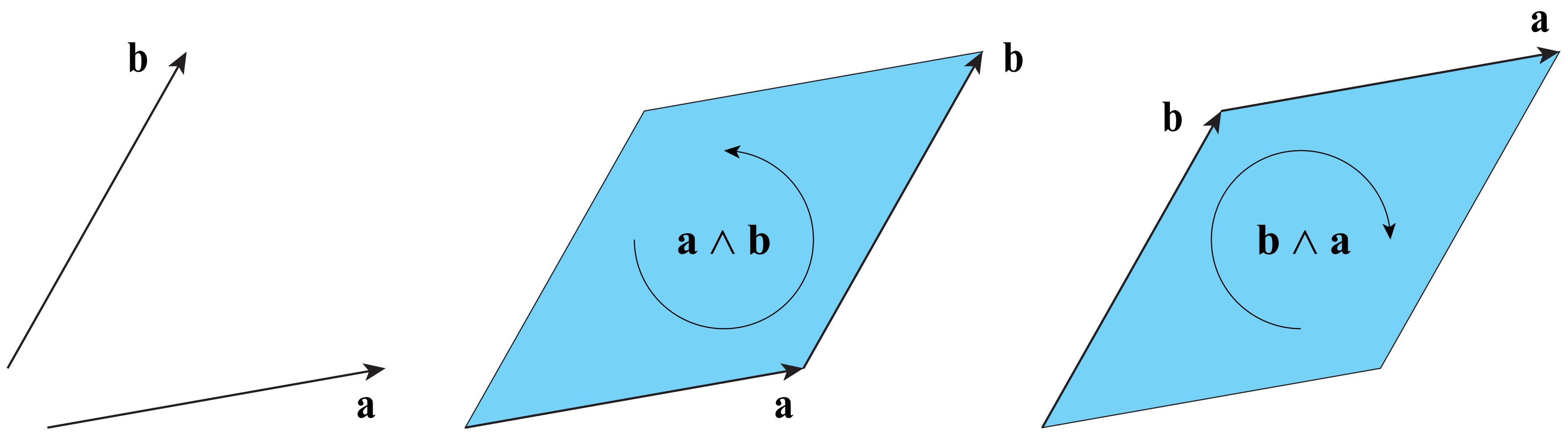
$$\mathbf{v} \wedge \mathbf{v} = 0$$

  - Antisymmetric on vectors:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

Bivectors

- Wedge product of two vectors **a** and **b**
- Produces a new type of object



Bivectors

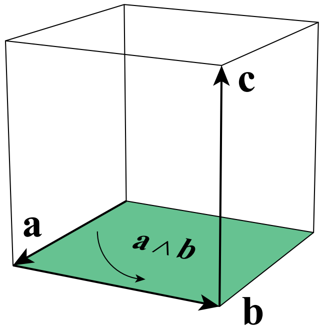
- Wedge product of two vectors **a** and **b**:

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b} &= \left(a_y b_z - a_z b_y\right)(\mathbf{e}_2\wedge\mathbf{e}_3)\\ &+ \left(a_z b_x - a_x b_z\right)(\mathbf{e}_3\wedge\mathbf{e}_1)\\ &+ \left(a_x b_y - a_y b_x\right)(\mathbf{e}_1\wedge\mathbf{e}_2)\end{aligned}
\qquad
\begin{aligned}\mathbf{a}\wedge\mathbf{b} &= \left(a_y b_z - a_z b_y\right)\mathbf{e}_{23}\\ &+ \left(a_z b_x - a_x b_z\right)\mathbf{e}_{31}\\ &+ \left(a_x b_y - a_y b_x\right)\mathbf{e}_{12}\end{aligned}$$

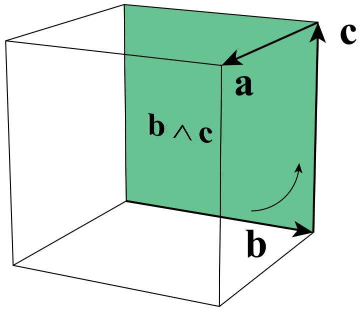
- Cross product appears!

Trivectors

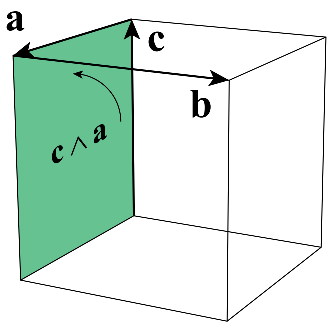
- Wedge product of three vectors **a**, **b**, and **c**



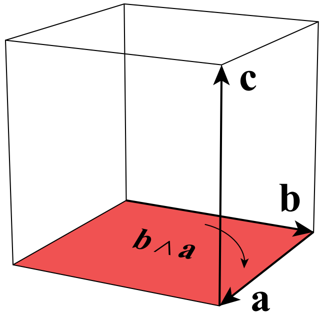
$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$



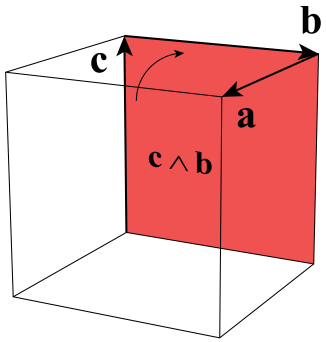
$\mathbf{b} \wedge \mathbf{c} \wedge \mathbf{a}$



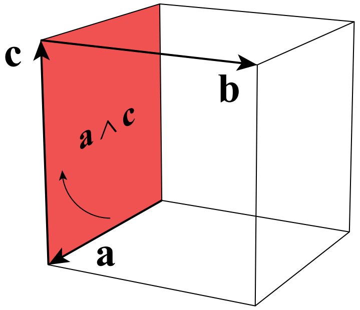
$\mathbf{c} \wedge \mathbf{a} \wedge \mathbf{b}$



$\mathbf{b} \wedge \mathbf{a} \wedge \mathbf{c}$



$\mathbf{c} \wedge \mathbf{b} \wedge \mathbf{a}$



$\mathbf{a} \wedge \mathbf{c} \wedge \mathbf{b}$

Trivectors

- Wedge product of three vectors **a**, **b**, and **c**

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \left(a_x b_y c_z + a_y b_z c_x + a_z b_x c_y - a_x b_z c_y - a_y b_x c_z - a_z b_y c_x \right) \mathbf{e}_{123}$$

- Determinant of 3 × 3 matrix with columns **a**, **b**, and **c**

Trivectors

- Wedge product of vector **a** and bivector **b**

$$\mathbf{a} = a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3$$

$$\mathbf{b} = b_x\mathbf{e}_{23} + b_y\mathbf{e}_{31} + b_z\mathbf{e}_{12}$$

$$\mathbf{a} \wedge \mathbf{b} = \left(a_x b_x + a_y b_y + a_z b_z\right)\mathbf{e}_{123}$$

- Dot product appears!

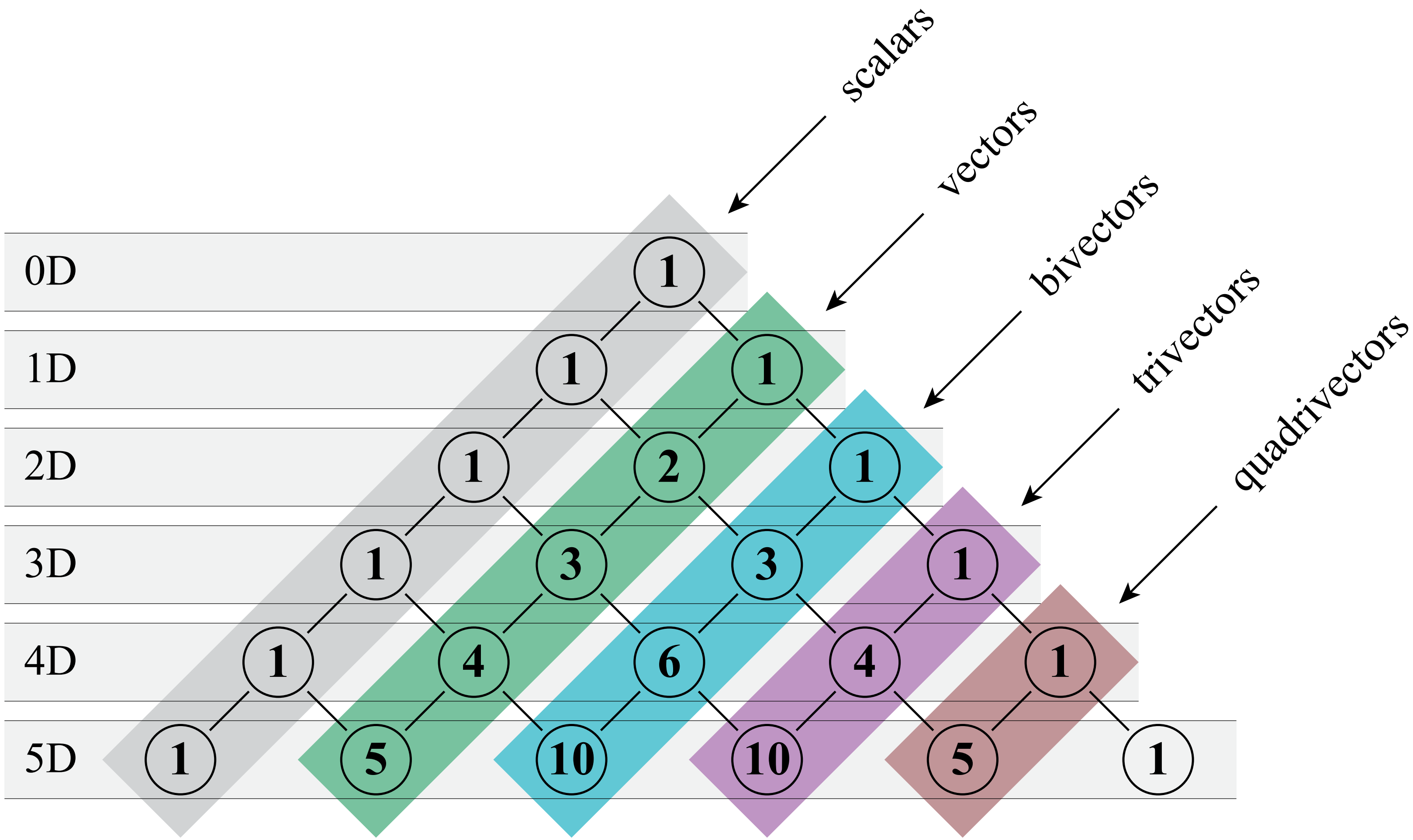
3D Vector Space

Scalars	s	Magnitudes
Vectors	$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$	Directed lengths

3D Exterior Algebra

Scalars	$s\mathbf{1}$	Magnitudes
Vectors	$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$	Directed lengths
Bivectors	$x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12}$	Directed areas
Trivectors	$t\mathbf{e}_{123}$	Directed volumes

Pascal’s Triangle



Rigid Exterior / Geometric Algebra

- Projective algebra with one extra dimension
- Contains points, lines, planes in 3D
- Can perform rotations, translations, screw transformations

4D Exterior Algebra

- Extends 4D vector space

- One scalar **1**
- Four vector basis elements
- Six bivector basis elements
- Four trivector basis elements
- One antiscalar $\mathbb{1}$

Type	Values	Grade / Antigrade	
Scalar	$\mathbf{1}$	0 / 4	▯▯▯▯
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4 = \mathbf{e}_n$	1 / 3	▮▯▯▯ ▯▮▯▯ ▯▯▮▯ ▯▯▯▮
Bivectors	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$	2 / 2	▮▯▯▮ ▯▮▯▮ ▯▯▮▮ ▯▮▮▯ ▮▯▮▯ ▮▮▯▯
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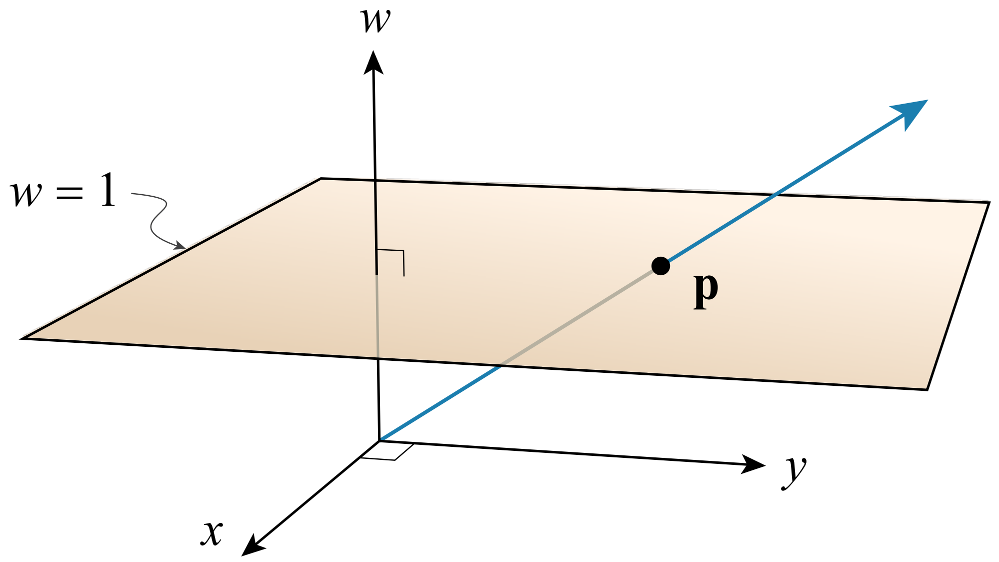
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	▮▮▮▮

Point

$$\mathbf{p} = \underbrace{p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3}_{\text{Position}} + \underbrace{p_w\mathbf{e}_4}_{\text{Weight}}$$



Sec. 2.4.1

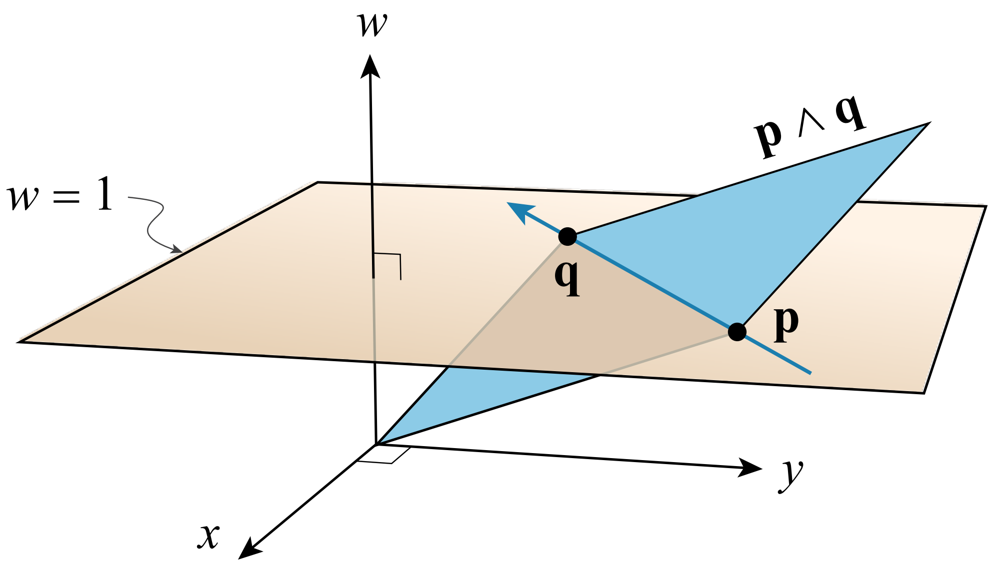
Special Points

- The origin is simply the point $\mathbf{e}_4$
- Point with zero weight lies at infinity in (x, y, z) direction
- Points at infinity in opposite directions are equivalent

Line

$$\mathbf{p} \wedge \mathbf{q} = (q_x p_w - p_x q_w)\mathbf{e}_{41} + (q_y p_w - p_y q_w)\mathbf{e}_{42} + (q_z p_w - p_z q_w)\mathbf{e}_{43}$$
$$+ (p_y q_z - p_z q_y)\mathbf{e}_{23} + (p_z q_x - p_x q_z)\mathbf{e}_{31} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$$

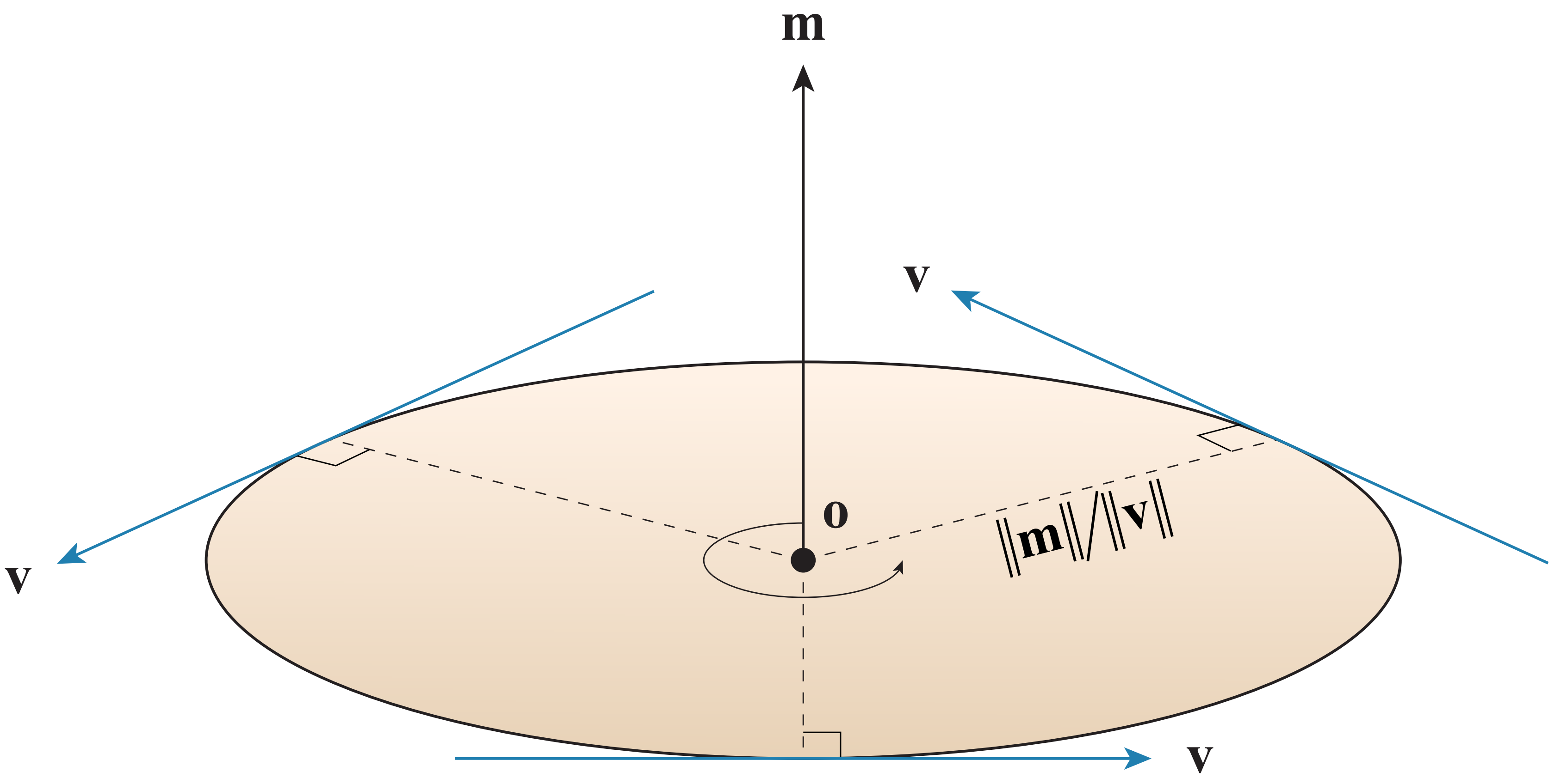
$$\boldsymbol{l} = \underbrace{l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}}_{\text{Moment}}$$



$$\boldsymbol{l}_\mathbf{v} \cdot \boldsymbol{l}_\mathbf{m} = 0$$

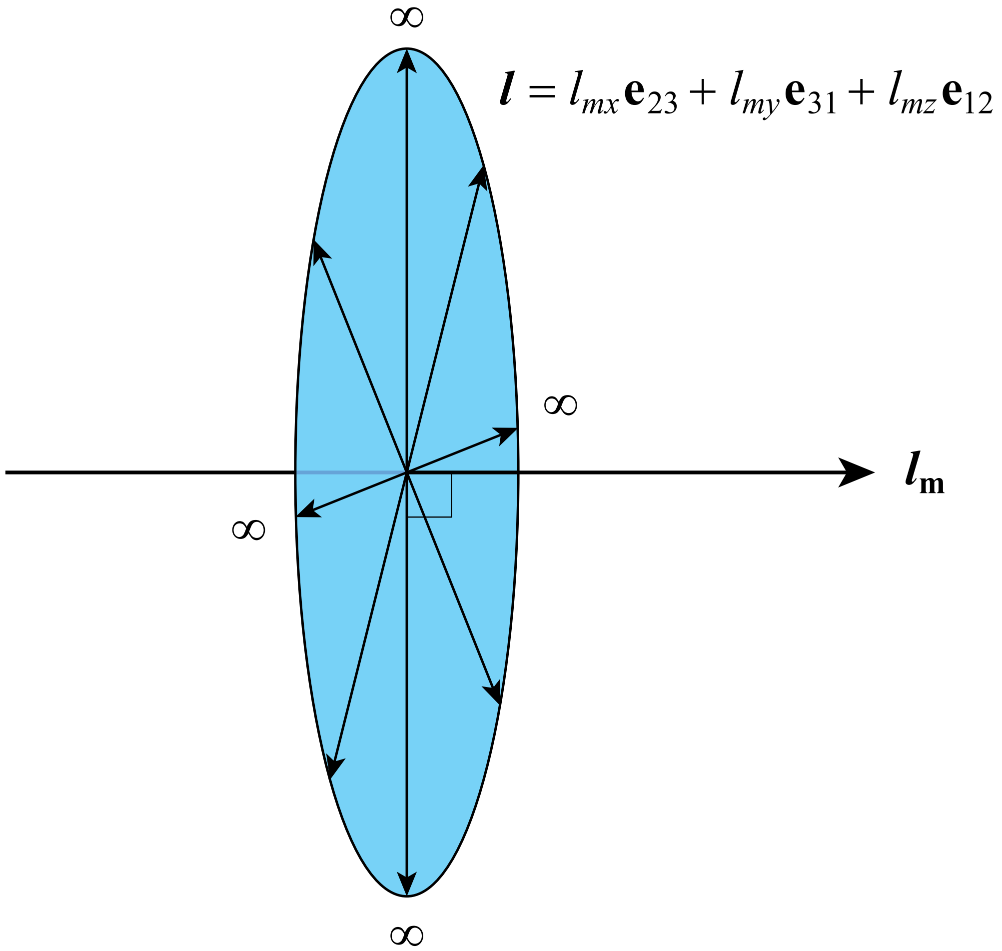
Line Moment

- Contains position information



Lines at Infinity

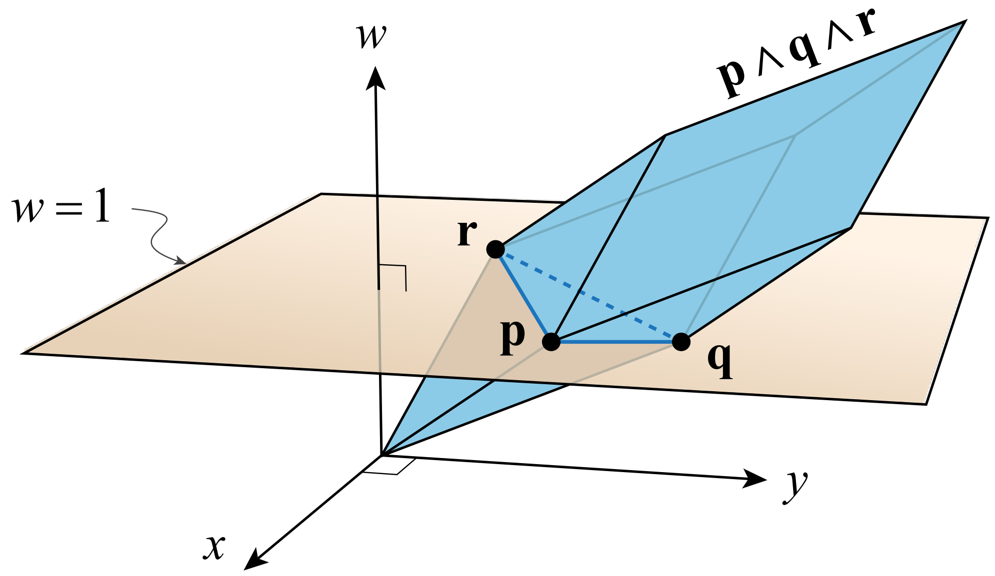
- Line with zero direction lies at infinity



Plane

$$\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx})\mathbf{e}_{423} + (l_{vz} p_x - l_{vx} p_z + l_{my})\mathbf{e}_{431}$$
$$+ (l_{vx} p_y - l_{vy} p_x + l_{mz})\mathbf{e}_{412} - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\mathbf{e}_{321}$$

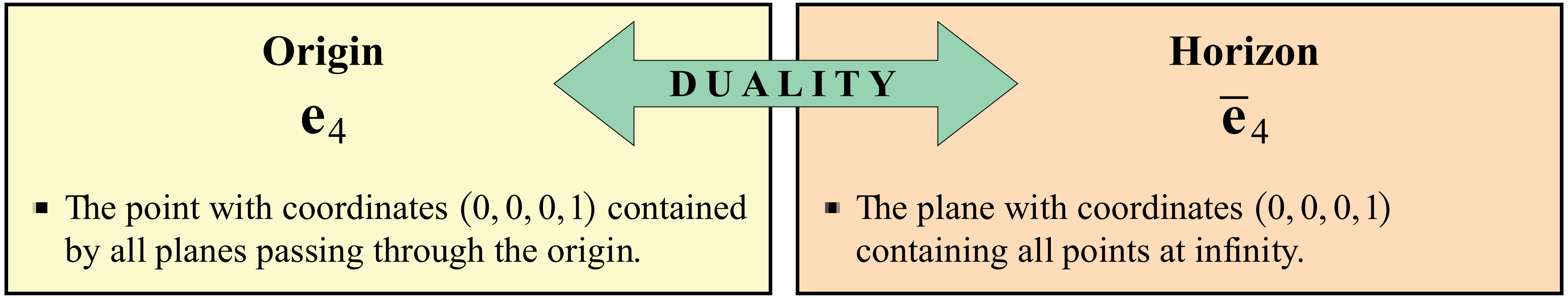
$$\mathbf{g} = \underbrace{g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w \mathbf{e}_{321}}_{\text{Position}}$$



Sec. 2.4.3

Horizon

- Plane with zero normal lies at infinity: $g_w \mathbf{e}_{321}$
- Contains all points at infinity, all lines at infinity
- Given special name *horizon*



Origin

$\mathbf{e}_4$

- The point with coordinates $(0, 0, 0, 1)$ contained by all planes passing through the origin.

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Horizon

$\overline{\mathbf{e}}_4$

- The plane with coordinates $(0, 0, 0, 1)$ containing all points at infinity.

4D Exterior Algebra

Scalars	$s\mathbf{1}$	Magnitudes
Vectors	$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_4$	Points
Bivectors	$v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	Lines
Trivectors	$g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321}$	Planes
Quadrivectors	$t\mathbb{1}$	Magnitudes

Complements

- Complement inverts full / empty dimensions
- Right complement denoted by overbar
- Left complement denoted by underbar
- For basis element **u**,

$$\mathbf{u} \wedge \overline{\mathbf{u}} = \mathbb{1} \qquad \underline{\mathbf{u}} \wedge \mathbf{u} = \mathbb{1}$$

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\overline{\mathbf{u}}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$\mathbf{1}$
$\underline{\mathbf{u}}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{1}$

Antiwedge Product

- Antiwedge product denoted by $\vee$

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Wedge Product

$\mathbf{a} \wedge \mathbf{b}$

- Combines the dimensions that are present in **a** and **b**.
- Result has grade $\operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b})$.
- Result has antigrade $\operatorname{ag}(\mathbf{a}) + \operatorname{ag}(\mathbf{b}) - n$.

Antiwedge Product

$\mathbf{a} \vee \mathbf{b}$

- Combines the dimensions that are absent in **a** and **b**.
- Result has grade $\operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b}) - n$.
- Result has antigrade $\operatorname{ag}(\mathbf{a}) + \operatorname{ag}(\mathbf{b})$.

De Morgan Laws

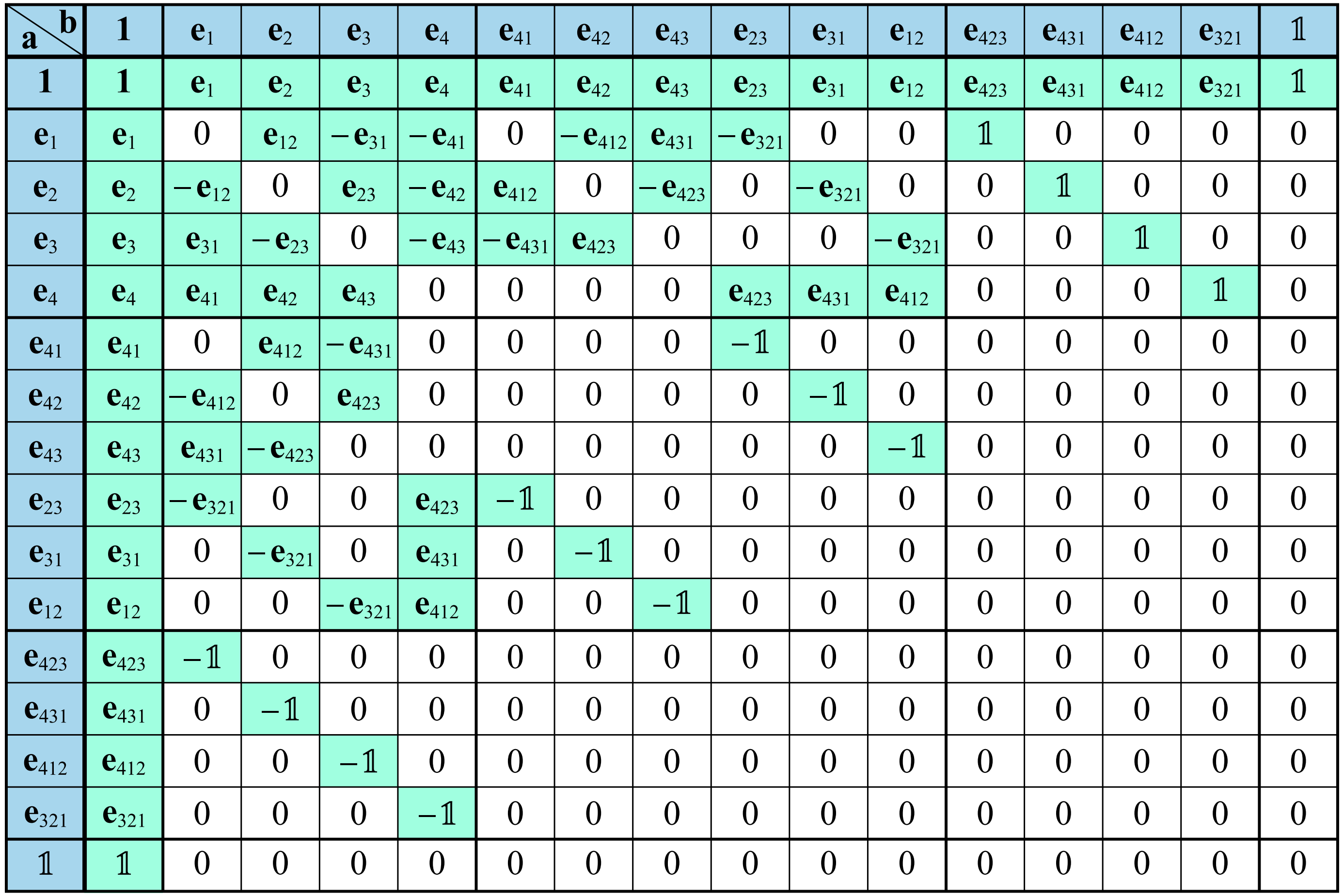
- Every operation with ‘anti’ in its name satisfies a De Morgan law:

$$\overline{\mathbf{a} \vee \mathbf{b}} = \overline{\mathbf{a}} \wedge \overline{\mathbf{b}} \qquad \underline{\mathbf{a} \vee \mathbf{b}} = \underline{\mathbf{a}} \wedge \underline{\mathbf{b}}$$

- To calculate anti-operation,
  - Take a complement of each input
  - Perform the regular operation
  - Take opposite complement of the result

4D Exterior Product

Wedge Product $\mathbf{a} \wedge \mathbf{b}$



$\mathbf{b}$ $\mathbf{a}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_1$	0	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	0	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0	0
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	0	$-\mathbf{e}_{423}$	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	0	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	0	0	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0
$\mathbf{e}_4$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	0	$\mathbf{e}_{423}$	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	0	0	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	0	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}$	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

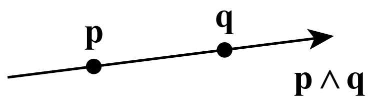
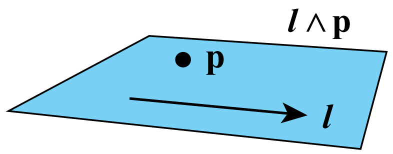
4D Exterior Antiproduct

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

$\mathbf{a}$ $\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	𝟙
$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	$\mathbf{e}_3$
$\mathbf{e}_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	$\mathbf{e}_4$
$\mathbf{e}_{41}$	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	0	0	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	0	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	0	0	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	0	$-\mathbf{1}$	0	0	0	0	0	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	0	0	$-\mathbf{1}$	0	0	0	0	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	0	0	0	$-\mathbf{e}_3$	$\mathbf{e}_2$	0	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	0	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_3$	0	$-\mathbf{e}_1$	$\mathbf{e}_{43}$	0	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	0	0	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$-\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
𝟙	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	𝟙

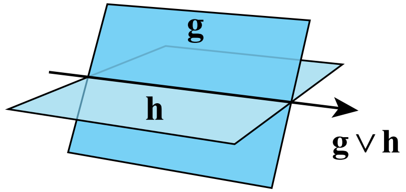
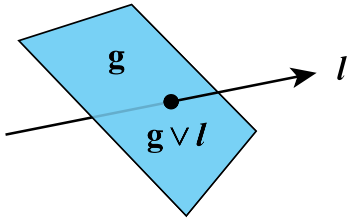
Join

- Wedge product performs join operation
- Produces higher-dimensional object containing both operands

Join Operation	**Illustration**
Line containing points **p** and **q**. $\mathbf{p} \wedge \mathbf{q} = (p_w q_x - p_x q_w)\mathbf{e}_{41} + (p_y q_z - p_z q_y)\mathbf{e}_{23}$ $+ (p_w q_y - p_y q_w)\mathbf{e}_{42} + (p_z q_x - p_x q_z)\mathbf{e}_{31}$ $+ (p_w q_z - p_z q_w)\mathbf{e}_{43} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$	
Plane containing line ***l*** and point **p**. $\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx} p_w)\mathbf{e}_{423}$ $+ (l_{vz} p_x - l_{vx} p_z + l_{my} p_w)\mathbf{e}_{431}$ $+ (l_{vx} p_y - l_{vy} p_x + l_{mz} p_w)\mathbf{e}_{412}$ $- (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\mathbf{e}_{321}$	

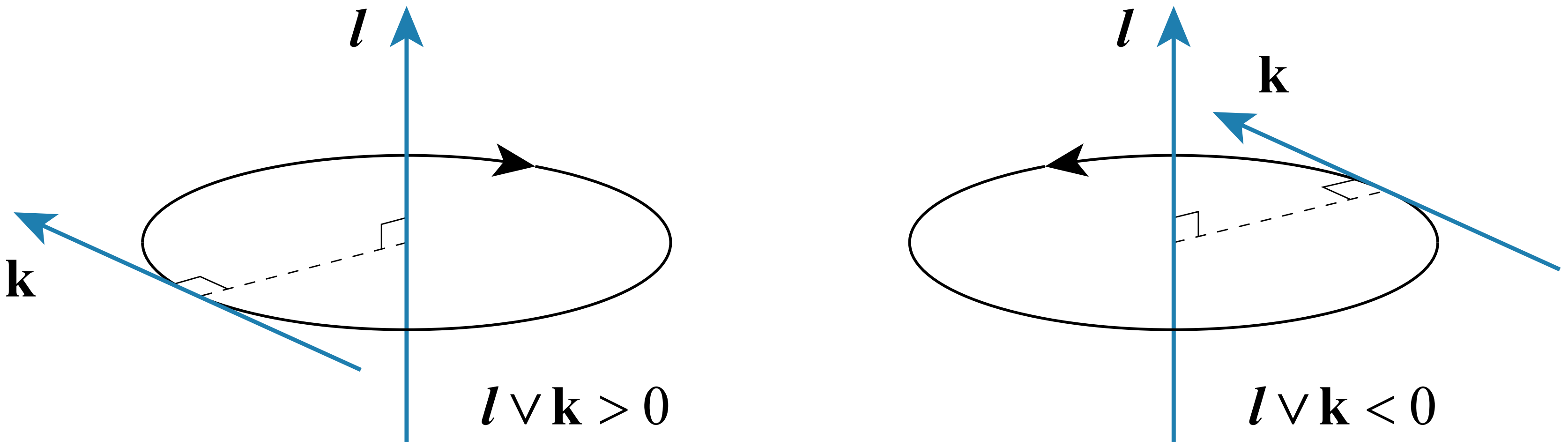
Meet

- Antiwedge product performs meet operation
- Produces lower-dimensional object at intersection of operands

Meet Operation	**Illustration**
Line where planes **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\mathbf{e}_{41} + (g_x h_w - g_w h_x)\mathbf{e}_{23}$ $+ (g_x h_z - g_z h_x)\mathbf{e}_{42} + (g_y h_w - g_w h_y)\mathbf{e}_{31}$ $+ (g_y h_x - g_x h_y)\mathbf{e}_{43} + (g_z h_w - g_w h_z)\mathbf{e}_{12}$	
Point where plane **g** and line ***l*** intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\mathbf{e}_1$ $+ (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\mathbf{e}_2$ $+ (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\mathbf{e}_3$ $- (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\mathbf{e}_4$	

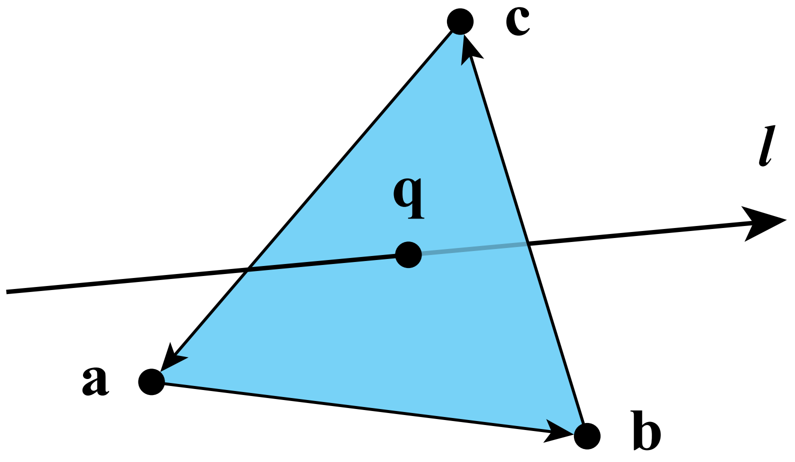
Line Crossing

- Sign of wedge product between lines gives crossing orientation



Line-Triangle Intersection

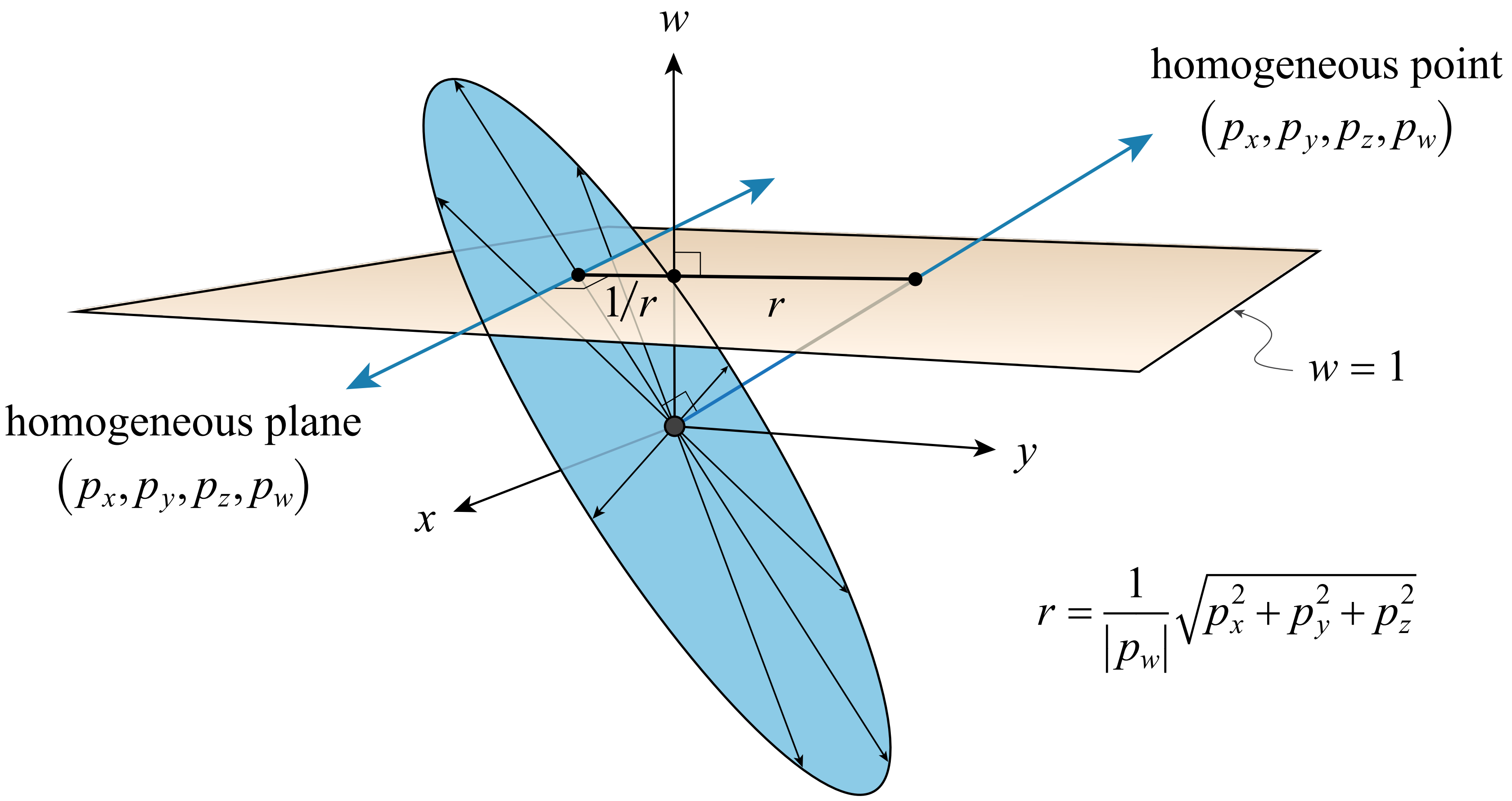
- Wedge product with all three edges of CCW-wound triangle must be positive



Duality

- Every object can be interpreted as two different things
- Every operation performs two different actions
- One interpretation corresponds to regular space
- The other interpretation corresponds to *antispace*

Duality



$$r = \frac{1}{|p_w|}\sqrt{p_x^2 + p_y^2 + p_z^2}$$

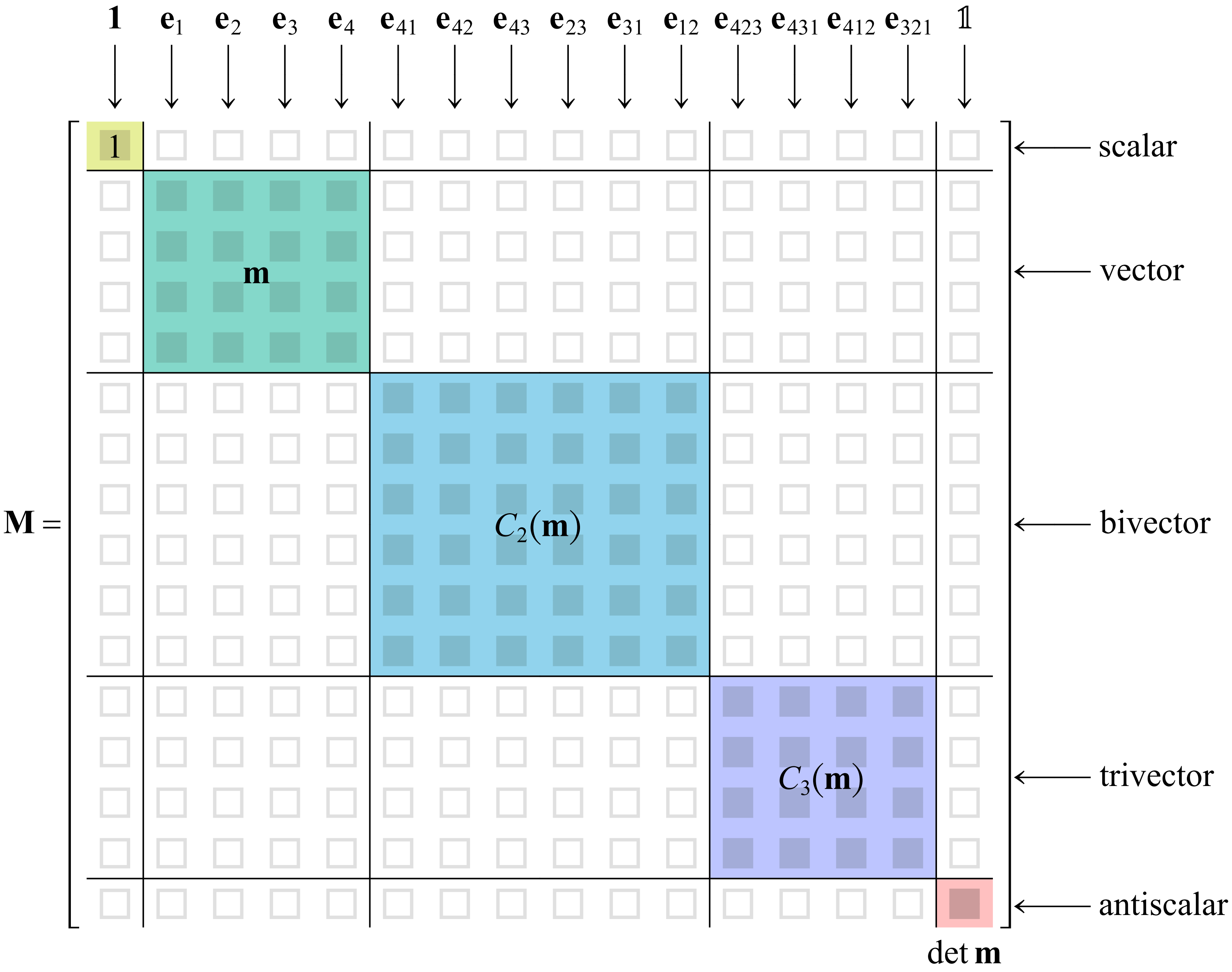
Exomorphisms

- Given an $n \times n$ linear transformation **m** that operates on vectors
- The exomorphism **M** is the $2^n \times 2^n$ matrix that operates on the whole algebra
- Exomorphism preserves structure under the wedge product:

$$\mathbf{M}(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{Ma}) \wedge (\mathbf{Mb})$$

Exomorphisms

- Matrix **M** is block diagonal
- Each block has columns given by wedge products of columns of the original matrix **m**
- These are called *compound matrices* of **m**


$\mathbf{1}$
$\mathbf{e}_1$
$\mathbf{e}_2$
$\mathbf{e}_3$
$\mathbf{e}_4$
$\mathbf{e}_{41}$
$\mathbf{e}_{42}$
$\mathbf{e}_{43}$
$\mathbf{e}_{23}$
$\mathbf{e}_{31}$
$\mathbf{e}_{12}$
$\mathbf{e}_{423}$
$\mathbf{e}_{431}$
$\mathbf{e}_{412}$
$\mathbf{e}_{321}$
$\mathbb{1}$
$\mathbf{M} =$
1
$\mathbf{m}$
$C_2(\mathbf{m})$
$C_3(\mathbf{m})$
$\det \mathbf{m}$
scalar
vector
bivector
trivector
antiscalar

Translation Exomorphism

$$\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -t_z & t_y & 1 & 0 & 0 \\ t_z & 0 & -t_x & 0 & 1 & 0 \\ -t_y & t_x & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_x & -t_y & -t_z & 1 \end{bmatrix}$$

Nonuniform Scale Exomorphism

$$\mathbf{m} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2(\mathbf{m}) = \begin{bmatrix} s_x & 0 & 0 & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 & 0 & 0 \\ 0 & 0 & s_z & 0 & 0 & 0 \\ 0 & 0 & 0 & s_y s_z & 0 & 0 \\ 0 & 0 & 0 & 0 & s_z s_x & 0 \\ 0 & 0 & 0 & 0 & 0 & s_x s_y \end{bmatrix}$$

$$C_3(\mathbf{m}) = \begin{bmatrix} s_y s_z & 0 & 0 & 0 \\ 0 & s_z s_x & 0 & 0 \\ 0 & 0 & s_x s_y & 0 \\ 0 & 0 & 0 & s_x s_y s_z \end{bmatrix}$$

The Metric Tensor

- $n \times n$ matrix that defines dot products of vectors

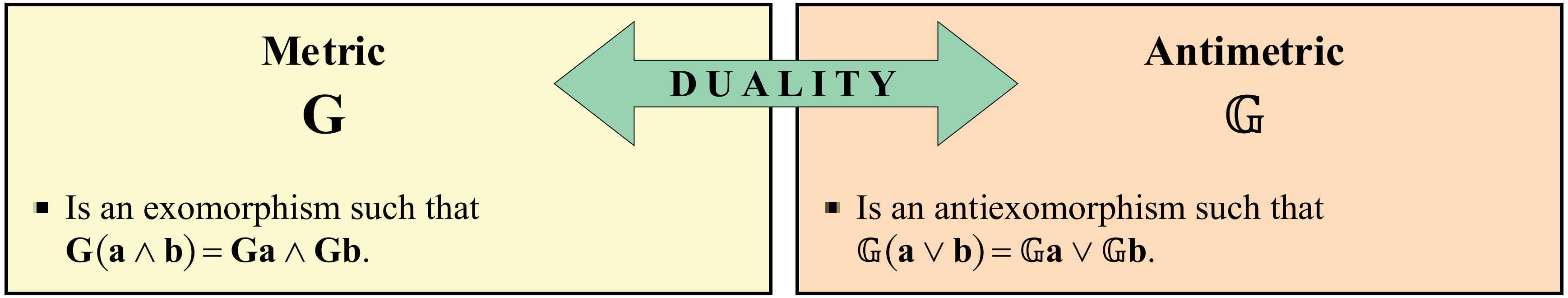
$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$
$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$
$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$
$$\mathbf{e}_4 \cdot \mathbf{e}_4 = 0$$

$$\mathfrak{g}_{ij} \equiv \mathbf{v}_i \cdot \mathbf{v}_j$$

Metric Exomorphism

- The metric tensor is a linear transformation
- It can be extended to a $2^n \times 2^n$ matrix **G** that applies to entire exterior algebra
- There is also an *antimetric* that satisfies $\mathbb{G}\mathbf{u} = \underline{\mathbf{G}\bar{\mathbf{u}}} = \overline{\mathbf{G}\underline{\mathbf{u}}}$



Metric

$\mathbf{G}$

- Is an exomorphism such that $\mathbf{G}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{G}\mathbf{a} \wedge \mathbf{G}\mathbf{b}$.

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Antimetric

$\mathbb{G}$

- Is an antiexomorphism such that $\mathbb{G}(\mathbf{a} \vee \mathbf{b}) = \mathbb{G}\mathbf{a} \vee \mathbb{G}\mathbf{b}$.

Metric and Antimetric

$$\mathbf{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0
\end{array}\right]$$

$$\mathbb{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1
\end{array}\right]$$

$$\mathbf{G}\mathbb{G} = \det(\mathfrak{g})\mathbf{I}$$

Bulk and Weight

- Bulk $\mathbf{u}_\bullet = \mathbf{G}\mathbf{u}$ All components without factor $\mathbf{e}_4$
- Weight $\mathbf{u}_\circ = \mathbb{G}\mathbf{u}$ All components with factor $\mathbf{e}_4$

Bulk

$\mathbf{u}_\bullet$

- Selects components of $\mathbf{u}$ without a factor of $\mathbf{e}_n$.
- Contains positional information about $\mathbf{u}$.
- Zero bulk means $\mathbf{u}$ contains the origin.

DUALITY

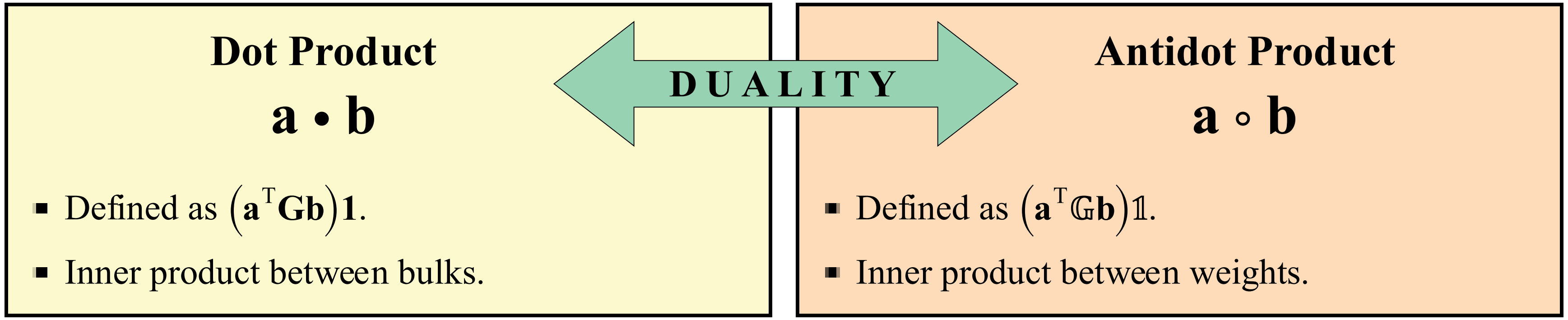
Weight

$\mathbf{u}_\circ$

- Selects components of $\mathbf{u}$ with a factor of $\mathbf{e}_n$.
- Contains directional information about $\mathbf{u}$.
- Zero weight means $\mathbf{u}$ is contained in the horizon.

Inner Products

- Dot product defined by metric: $\mathbf{a} \bullet \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}}\mathbf{G}\mathbf{b}\right)\mathbf{1}$
- Antidot product defined by antimetric: $\mathbf{a} \circ \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}}\mathbb{G}\mathbf{b}\right)\mathbb{1}$
- Satisfies De Morgan law: $\mathbf{a} \circ \mathbf{b} = \overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}}$



Dot Product

$\mathbf{a} \bullet \mathbf{b}$

- Defined as $\left(\mathbf{a}^{\mathrm{T}}\mathbf{G}\mathbf{b}\right)\mathbf{1}$.
- Inner product between bulks.

DUALITY

Antidot Product

$\mathbf{a} \circ \mathbf{b}$

- Defined as $\left(\mathbf{a}^{\mathrm{T}}\mathbb{G}\mathbf{b}\right)\mathbb{1}$.
- Inner product between weights.

Bulk and Weight Norms

- Two dot products induce two norms
- Bulk norm: $\left\|\mathbf{u}\right\|_{\bullet} = \sqrt{\mathbf{u} \bullet \mathbf{u}}$
- Weight norm: $\left\|\mathbf{u}\right\|_{\circ} = \sqrt{\mathbf{u} \circ \mathbf{u}}$
- Can generally have arbitrary values for same geometry due to homogeneity

Bulk and Weight Norms

Type	Bulk Norm	Weight Norm
Point **p**	$\Vert\mathbf{p}\Vert_\bullet = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2}$	$\Vert\mathbf{p}\Vert_\circ = \lvert p_w\rvert\mathbb{1}$
Line ***l***	$\Vert\boldsymbol{l}\Vert_\bullet = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$	$\Vert\boldsymbol{l}\Vert_\circ = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$
Plane **g**	$\Vert\mathbf{g}\Vert_\bullet = \lvert g_w\rvert\mathbf{1}$	$\Vert\mathbf{g}\Vert_\circ = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$

Unitization

- An object is *unitized* when its weight has magnitude one

Type	**Definition**	**Unitization**
Point **p**	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	$p_w^2 = 1$
Line ***l***	$\boldsymbol{l} = l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43} + l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}$	$l_{vx}^2 + l_{vy}^2 + l_{vz}^2 = 1$
Plane **g**	$\mathbf{g} = g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321}$	$g_x^2 + g_y^2 + g_z^2 = 1$

Geometric Norm

- Bulk and weight norms by themselves not very meaningful
- But add them, and result is a *homogeneous magnitude*
- Represents distance from origin
- Called the *geometric norm*

$$\|\mathbf{u}\| = \|\mathbf{u}\|_{\bullet} + \|\mathbf{u}\|_{\bigcirc} = \sqrt{\mathbf{u} \bullet \mathbf{u}} + \sqrt{\mathbf{u} \circ \mathbf{u}}$$

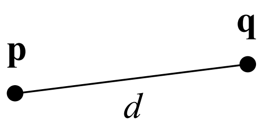
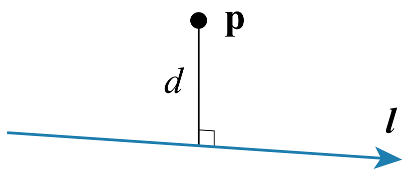
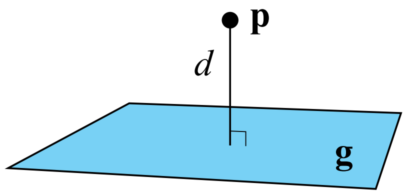
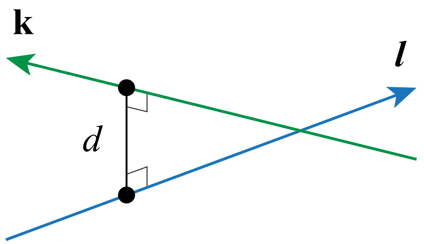
- Two-component quantity, sum of scalar and antiscalar $s\mathbf{1} + t\mathbb{1}$
- Can be unitized by making weight one

Sec. 2.10.3

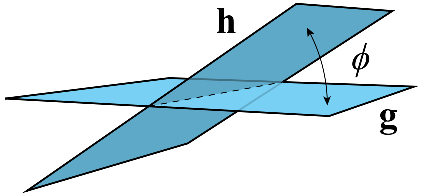
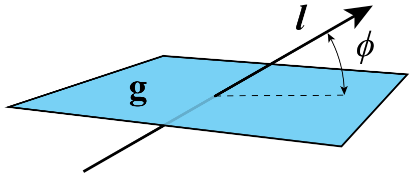
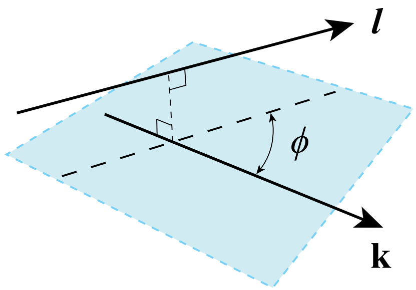
Geometric Norm

Type	**Geometric Norm**	**Interpretation**
Point $\mathbf{p}$	$\widehat{\lVert\mathbf{p}\rVert} = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from the origin to the point $\mathbf{p}$.
Line $\boldsymbol{l}$	$\widehat{\lVert\boldsymbol{l}\rVert} = \dfrac{\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}}{\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$	Perpendicular distance from the origin to the line $\boldsymbol{l}$.
Plane $\mathbf{g}$	$\widehat{\lVert\mathbf{g}\rVert} = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$	Perpendicular distance from the origin to the plane $\mathbf{g}$.

Euclidean Distance

Distance Formula	Illustration
Distance d between points $\mathbf{p}$ and $\mathbf{q}$. $d(\mathbf{p}, \mathbf{q}) = \lVert \mathbf{q}_{xyz} p_w - \mathbf{p}_{xyz} q_w \rVert \mathbf{1} + \lvert p_w q_w \rvert \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and line $\boldsymbol{l}$. $d(\mathbf{p}, \boldsymbol{l}) = \lVert \boldsymbol{l}_\mathbf{v} \times \mathbf{p}_{xyz} + p_w \boldsymbol{l}_\mathbf{m} \rVert \mathbf{1} + \lVert p_w \boldsymbol{l}_\mathbf{v} \rVert \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and plane $\mathbf{g}$. $d(\mathbf{p}, \mathbf{g}) = (\mathbf{p} \cdot \mathbf{g}) \mathbf{1} + \lVert p_w \mathbf{g}_{xyz} \rVert \mathbb{1}$	
Perpendicular distance d between skew lines $\boldsymbol{l}$ and $\mathbf{k}$. $d(\boldsymbol{l}, \mathbf{k}) = -(\boldsymbol{l}_\mathbf{v} \cdot \mathbf{k}_\mathbf{m} + \boldsymbol{l}_\mathbf{m} \cdot \mathbf{k}_\mathbf{v}) \mathbf{1} + \lVert \boldsymbol{l}_\mathbf{v} \times \mathbf{k}_\mathbf{v} \rVert \mathbb{1}$	

Euclidean Angle

Angle Formula	Illustration
Cosine of angle ϕ between planes **g** and **h**. $\cos\phi(\mathbf{g},\mathbf{h}) = \left(\mathbf{g}_{xyz}\cdot\mathbf{h}_{xyz}\right)\mathbf{1} + \lVert\mathbf{g}\rVert_{\bigcirc}\lVert\mathbf{h}\rVert_{\bigcirc}$	
Cosine of angle ϕ between plane **g** and line ***l***. $\cos\phi(\mathbf{g},\boldsymbol{l}) = \lVert\mathbf{g}_{xyz}\times\boldsymbol{l}_{\mathbf{v}}\rVert\mathbf{1} + \lVert\mathbf{g}\rVert_{\bigcirc}\lVert\boldsymbol{l}\rVert_{\bigcirc}$	
Cosine of angle ϕ between lines ***l*** and **k**. $\cos\phi(\boldsymbol{l},\mathbf{k}) = (\boldsymbol{l}_{\mathbf{v}}\cdot\mathbf{k}_{\mathbf{v}})\mathbf{1} + \lVert\boldsymbol{l}\rVert_{\bigcirc}\lVert\mathbf{k}\rVert_{\bigcirc}$	

Bulk and Weight Duals

- Multiply by metric or antimetric, then take complement

Bulk Dual

$\mathbf{u}^{\bigstar}$

- Defined as $\overline{\mathbf{Gu}}$.
- Gives the dual of the bulk components of **u**.

DUALITY

Weight Dual

$\mathbf{u}^{\text{☆}}$

- Defined as $\overline{\mathbb{G}\mathbf{u}}$.
- Gives the dual of the weight components of **u**.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{u}^{\bigstar}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0
$\mathbf{u}_{\bigstar}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$\mathbf{e}_4$	0
$\mathbf{u}^{\text{☆}}$	0	0	0	0	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	0	$\mathbf{1}$
$\mathbf{u}_{\text{☆}}$	0	0	0	0	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	$\mathbf{1}$

Hodge Dual

- Right bulk dual is equivalent to Hodge dual

$$\mathbf{u}^{\bigstar} = \overline{\mathbf{G}\mathbf{u}}$$

- For **a** and **b** with same grade,

$$\mathbf{a} \wedge \mathbf{b}^{\bigstar} = (\mathbf{a} \bullet \mathbf{b})\mathbb{1}$$

Interior Products

- Two exterior products combined with two duals
- Eight *interior* products using right and left duals

• Bulk contraction	$\mathbf{a} \vee \mathbf{b}^{\bigstar}$	$\mathbf{b}_{\bigstar} \vee \mathbf{a}$
• Weight contraction	$\mathbf{a} \vee \mathbf{b}^{\medwhitestar}$	$\mathbf{b}_{\medwhitestar} \vee \mathbf{a}$
• Bulk expansion	$\mathbf{a} \wedge \mathbf{b}^{\bigstar}$	$\mathbf{b}_{\bigstar} \wedge \mathbf{a}$
• Weight expansion	$\mathbf{a} \wedge \mathbf{b}^{\medwhitestar}$	$\mathbf{b}_{\medwhitestar} \wedge \mathbf{a}$

Interior Products

- Right and left interior products differ by grade-dependent sign:

$$\mathbf{b}_{*} \vee \mathbf{a} = (-1)^{\operatorname{gr}(\mathbf{b})[\operatorname{gr}(\mathbf{a})+\operatorname{gr}(\mathbf{b})]}\mathbf{a} \vee \mathbf{b}^{*}$$

$$\mathbf{b}_{*} \wedge \mathbf{a} = (-1)^{\operatorname{ag}(\mathbf{b})[\operatorname{ag}(\mathbf{a})+\operatorname{ag}(\mathbf{b})]}\mathbf{a} \wedge \mathbf{b}^{*}$$

- Here, $*$ is either ★ or ☆

- Really need only four interior products

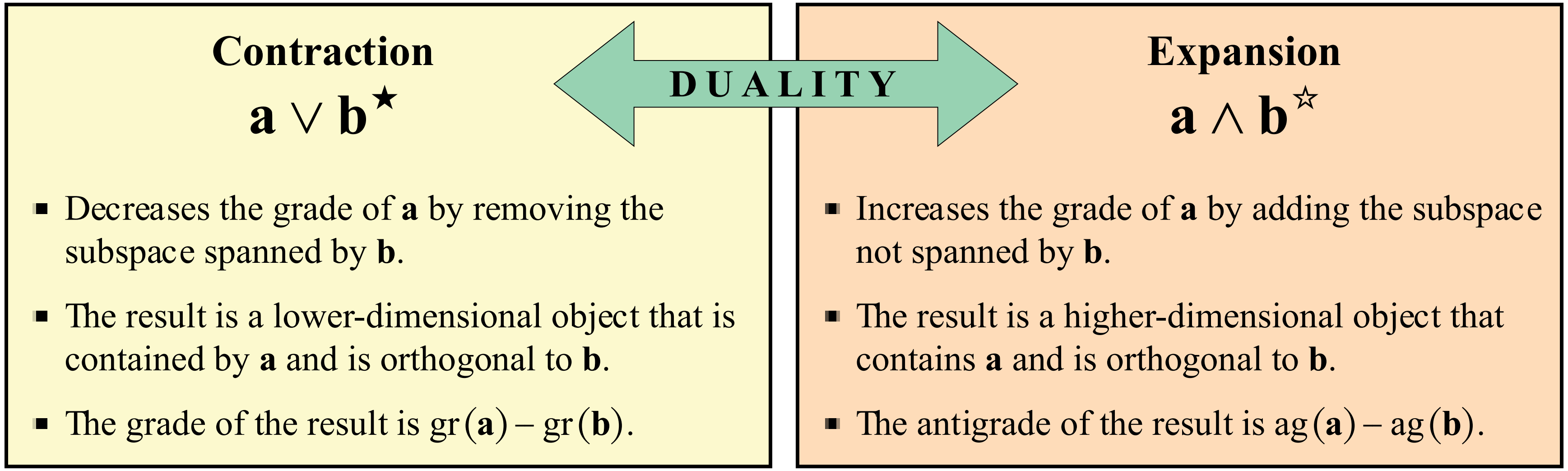
Interior Products

- Interior products reduce to inner products for same grade:

$$\begin{aligned}
\mathbf{a} \vee \mathbf{b}^{\bigstar} &= \mathbf{a} \bullet \mathbf{b}, &\qquad& \text{when } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b}) \\
\mathbf{a} \vee \mathbf{b}^{\text{☆}} &= (\mathbf{a} \circ \mathbf{b}) \vee \mathbf{1}, &\qquad& \text{when } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b}) \\
\mathbf{a} \wedge \mathbf{b}^{\text{☆}} &= \mathbf{a} \circ \mathbf{b}, &\qquad& \text{when } \operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b}) \\
\mathbf{a} \wedge \mathbf{b}^{\bigstar} &= (\mathbf{a} \bullet \mathbf{b}) \wedge \mathbb{1}, &\qquad& \text{when } \operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b})
\end{aligned}$$

Contraction and Expansion

- Subtract grades or antigrades



Contraction

$\mathbf{a} \vee \mathbf{b}^{\bigstar}$

- Decreases the grade of **a** by removing the subspace spanned by **b**.
- The result is a lower-dimensional object that is contained by **a** and is orthogonal to **b**.
- The grade of the result is $\operatorname{gr}(\mathbf{a}) - \operatorname{gr}(\mathbf{b})$.

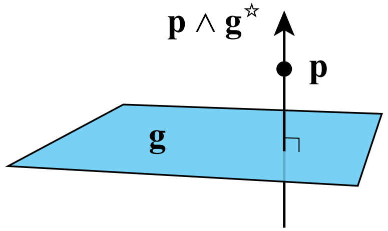
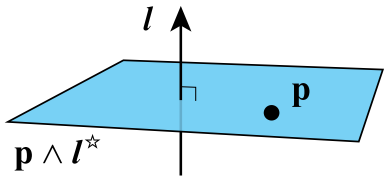
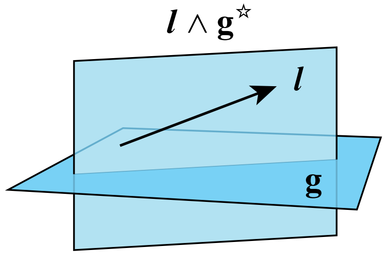
DUALITY

Expansion

$\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$

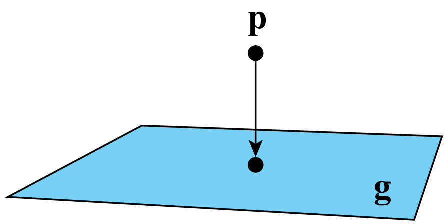
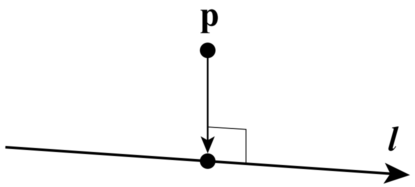
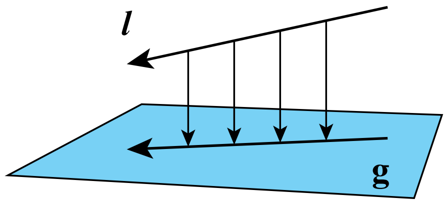
- Increases the grade of **a** by adding the subspace not spanned by **b**.
- The result is a higher-dimensional object that contains **a** and is orthogonal to **b**.
- The antigrade of the result is $\operatorname{ag}(\mathbf{a}) - \operatorname{ag}(\mathbf{b})$.

Weight Expansion

Expansion Operation	Illustration
Line containing point **p** and orthogonal to plane **g**. $\mathbf{p} \wedge \mathbf{g}^{\star} = -p_w g_x \mathbf{e}_{41} + (p_z g_y - p_y g_z)\mathbf{e}_{23}$ $- p_w g_y \mathbf{e}_{42} + (p_x g_z - p_z g_x)\mathbf{e}_{31}$ $- p_w g_z \mathbf{e}_{43} + (p_y g_x - p_x g_y)\mathbf{e}_{12}$	
Plane containing point **p** and orthogonal to line ***l***. $\mathbf{p} \wedge \boldsymbol{l}^{\star} = -p_w l_{vx} \mathbf{e}_{423} - p_w l_{vy} \mathbf{e}_{431} - p_w l_{vz} \mathbf{e}_{412}$ $+ (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\mathbf{e}_{321}$	
Plane containing line ***l*** and orthogonal to plane **g**. $\boldsymbol{l} \wedge \mathbf{g}^{\star} = (l_{vy} g_z - l_{vz} g_y)\mathbf{e}_{423}$ $+ (l_{vz} g_x - l_{vx} g_z)\mathbf{e}_{431}$ $+ (l_{vx} g_y - l_{vy} g_x)\mathbf{e}_{412}$ $- (l_{mx} g_x + l_{my} g_y + l_{mz} g_z)\mathbf{e}_{321}$	

Sec. 2.13.5

Orthogonal Projection

Projection Operation	**Illustration**
Orthogonal projection of point **p** onto plane **g**. $\mathbf{g} \vee \left(\mathbf{p} \wedge \mathbf{g}^{\star}\right) = \left(g_x^2 + g_y^2 + g_z^2\right)\left(p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4\right)$ $- \left(g_x p_x + g_y p_y + g_z p_z + g_w p_w\right)\left(g_x \mathbf{e}_1 + g_y \mathbf{e}_2 + g_z \mathbf{e}_3\right)$	
Orthogonal projection of point **p** onto line ***l***. $\boldsymbol{l} \vee \left(\mathbf{p} \wedge \boldsymbol{l}^{\star}\right) = \left(l_{vx} p_x + l_{vy} p_y + l_{vz} p_z\right)\left(l_{vx} \mathbf{e}_1 + l_{vy} \mathbf{e}_2 + l_{vz} \mathbf{e}_3\right)$ $+ \left(l_{vy} l_{mz} - l_{vz} l_{my}\right) p_w \mathbf{e}_1 + \left(l_{vz} l_{mx} - l_{vx} l_{mz}\right) p_w \mathbf{e}_2$ $+ \left(l_{vx} l_{my} - l_{vy} l_{mx}\right) p_w \mathbf{e}_3 + \left(l_{vx}^2 + l_{vy}^2 + l_{vz}^2\right) p_w \mathbf{e}_4$	
Orthogonal projection of line ***l*** onto plane **g**. $\mathbf{g} \vee \left(\boldsymbol{l} \wedge \mathbf{g}^{\star}\right) = \left(g_x^2 + g_y^2 + g_z^2\right)\left(l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43}\right)$ $- \left(g_x l_{vx} + g_y l_{vy} + g_z l_{vz}\right)\left(g_x \mathbf{e}_{41} + g_y \mathbf{e}_{42} + g_z \mathbf{e}_{43}\right)$ $+ \left(g_x l_{mx} + g_y l_{my} + g_z l_{mz}\right)\left(g_x \mathbf{e}_{23} + g_y \mathbf{e}_{31} + g_z \mathbf{e}_{12}\right)$ $+ \left(g_z l_{vy} - g_y l_{vz}\right) g_w \mathbf{e}_{23} + \left(g_x l_{vz} - g_z l_{vx}\right) g_w \mathbf{e}_{31}$ $+ \left(g_y l_{vx} - g_x l_{vy}\right) g_w \mathbf{e}_{12}$	

Support

- Orthogonal projection of origin onto line or plane
- Support is point closest to origin contained by object

$$\operatorname{sup}(\boldsymbol{l}) = \left(l_{vy}l_{mz} - l_{vz}l_{my}\right)\mathbf{e}_1 + \left(l_{vz}l_{mx} - l_{vx}l_{mz}\right)\mathbf{e}_2 + \left(l_{vx}l_{my} - l_{vy}l_{mx}\right)\mathbf{e}_3 + \boldsymbol{l}_{\mathbf{v}}^2\mathbf{e}_4$$

$$\operatorname{sup}(\mathbf{g}) = -g_x g_w \mathbf{e}_1 - g_y g_w \mathbf{e}_2 - g_z g_w \mathbf{e}_3 + \left(g_x^2 + g_y^2 + g_z^2\right)\mathbf{e}_4$$

Geometric / Clifford Algebra

- Geometric product $\mathbf{a} ⟑ \mathbf{b}$

- Geometric antiproduct $\mathbf{a} ⟇ \mathbf{b}$

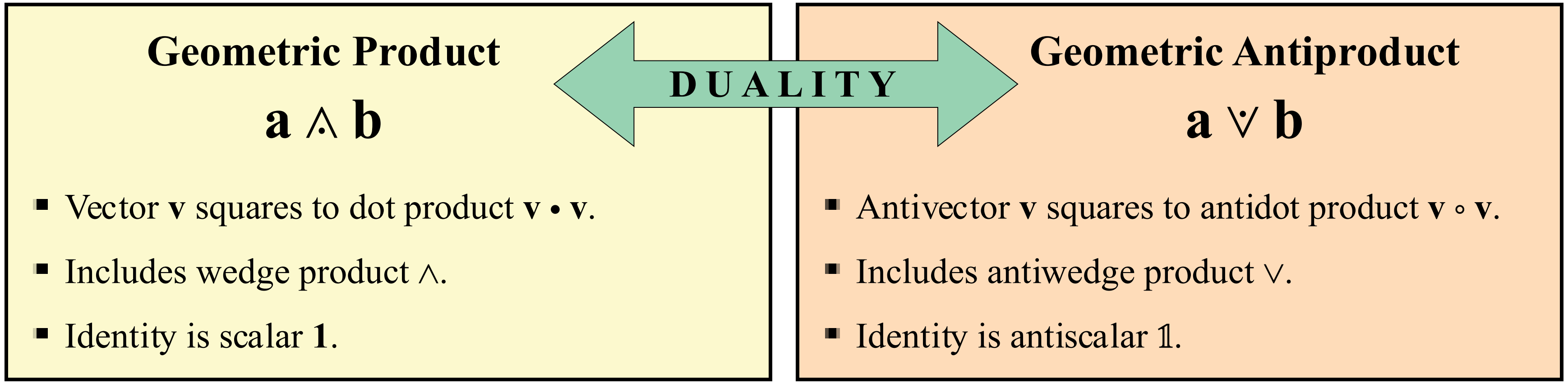
- We use upward and downward wedge with dot inside
- “Wedge-dot” and “Antiwedge-dot”

- G.P. historically denoted by juxtaposition without symbol
- But duality gives us two products that need distinguishing

Geometric Product and Antiproduct

- Vectors square to inner product instead of zero
- Product satisfy the usual De Morgan law

$$\mathbf{a} \mathbin{\dot{\vee}} \mathbf{b} = \overline{\underline{\mathbf{a}} \mathbin{\dot{\wedge}} \underline{\mathbf{b}}}$$



Geometric Product

$\mathbf{a} \mathbin{\dot{\wedge}} \mathbf{b}$

- Vector $\mathbf{v}$ squares to dot product $\mathbf{v} \bullet \mathbf{v}$.
- Includes wedge product $\wedge$.
- Identity is scalar $\mathbf{1}$.

DUALITY

Geometric Antiproduct

$\mathbf{a} \mathbin{\dot{\vee}} \mathbf{b}$

- Antivector $\mathbf{v}$ squares to antidot product $\mathbf{v} \circ \mathbf{v}$.
- Includes antiwedge product $\vee$.
- Identity is antiscalar $\mathbb{1}$.

Geometric Products

- For <u>vectors</u> **a** and **b**:

$$\mathbf{a} ⟑ \mathbf{b} = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \bullet \mathbf{b}$$

- For <u>antivectors</u> **a** and **b**:

$$\mathbf{a} ⟇ \mathbf{b} = \mathbf{a} \vee \mathbf{b} + \mathbf{a} \circ \mathbf{b}$$

Geometric Products

- For vector **a** and arbitrary **B**:

$$\mathbf{a} \mathbin{\underset{\cdot}{\wedge}} \mathbf{B} = \mathbf{a} \wedge \mathbf{B} + \mathbf{B} \vee \mathbf{a}^{\bigstar}$$

- For antivector **a** and arbitrary **B**:

$$\mathbf{a} \mathbin{\overset{\cdot}{\vee}} \mathbf{B} = \mathbf{a} \vee \mathbf{B} + \mathbf{B} \wedge \mathbf{a}^{\text{☆}}$$

Geometric Products

- In general, there are more terms for **A** and **B** with higher grades

- In 4D algebra, arbitrary **A** and **B** multiply as

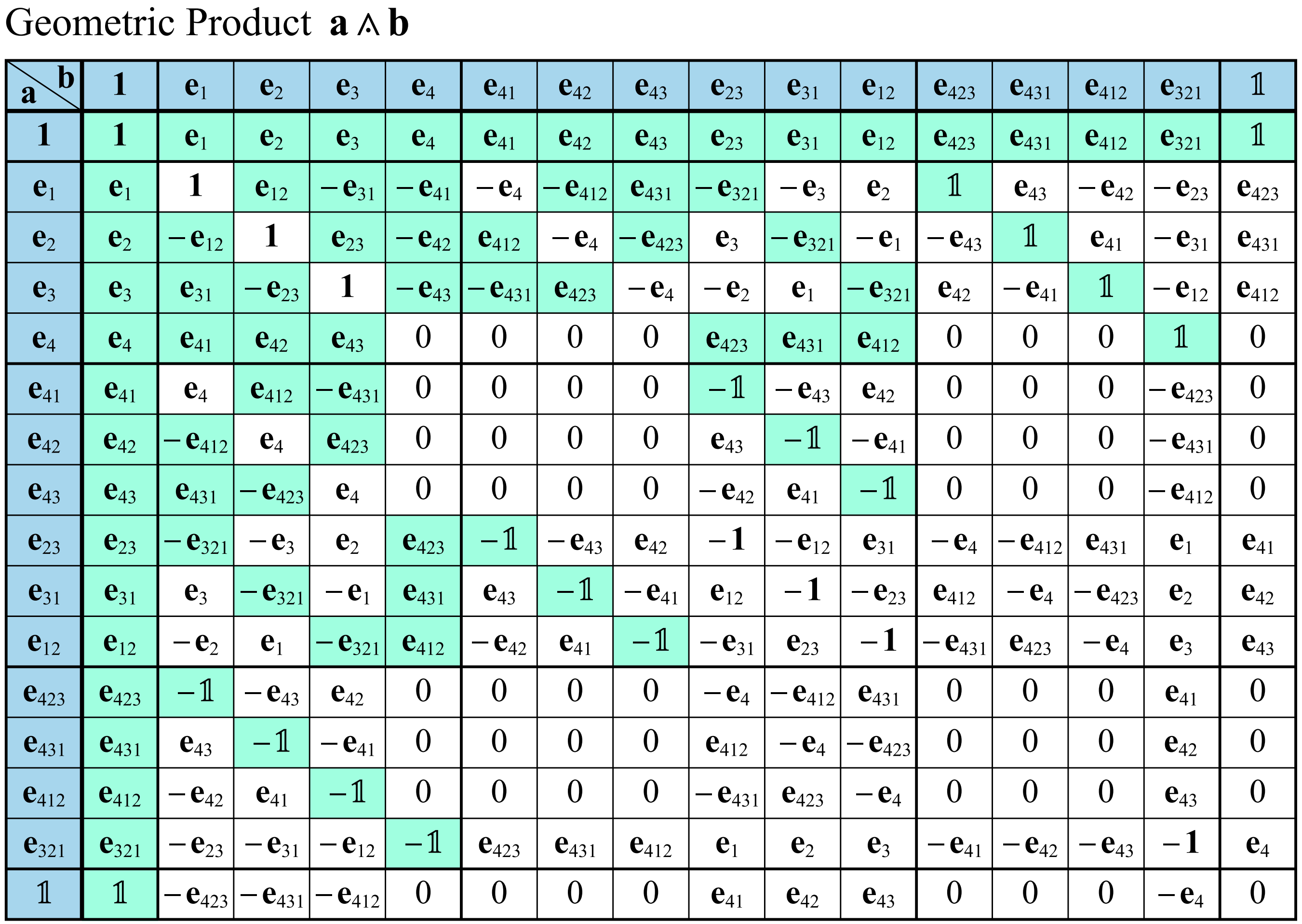
$$\mathbf{A} \mathbin{\dot\wedge} \mathbf{B} = \mathbf{A} \wedge \mathbf{B} + \mathbf{A} \times \mathbf{B} + \mathbf{A} \bullet \tilde{\mathbf{B}}$$

$$\uparrow$$

commutator product

$$\mathbf{A} \times \mathbf{B} = \tfrac{1}{2}(\mathbf{A} \mathbin{\dot\wedge} \mathbf{B} - \mathbf{B} \mathbin{\dot\wedge} \mathbf{A})$$

4D Geometric Product



Geometric Product $\mathbf{a} \mathbin{⟑} \mathbf{b}$

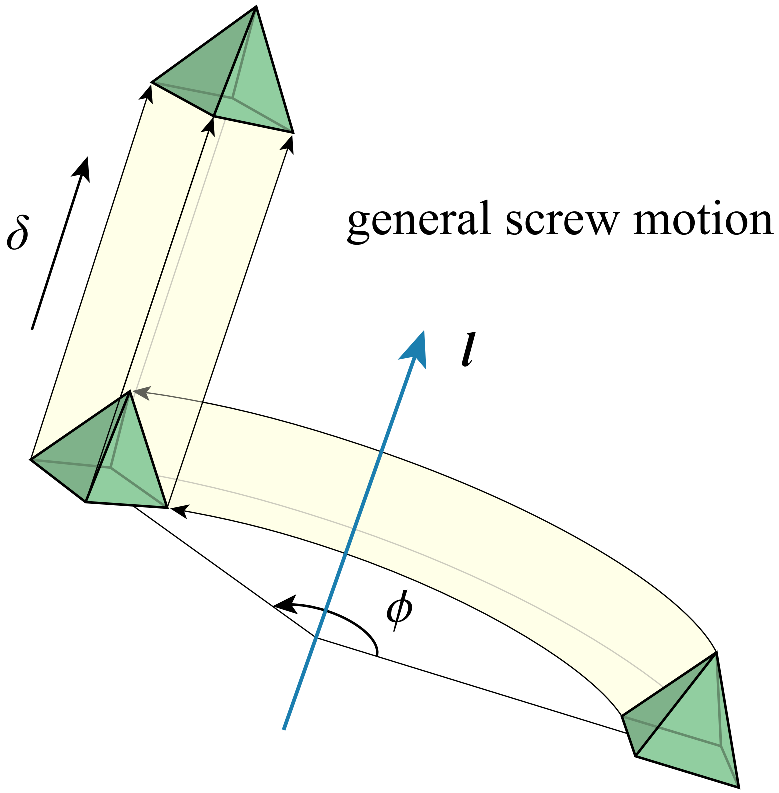
$\mathbf{a}$ $\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_{1}$	$\mathbf{e}_{1}$	$\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{2}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	$\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{43}$	$\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{3}$	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{1}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbb{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{4}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	$-\mathbf{e}_{423}$	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{4}$	$\mathbf{e}_{423}$	0	0	0	0	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	$-\mathbf{e}_{431}$	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{4}$	0	0	0	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	$-\mathbf{e}_{412}$	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$-\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_{1}$	$\mathbf{e}_{41}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	$-\mathbf{1}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{2}$	$\mathbf{e}_{42}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{3}$	$\mathbf{e}_{43}$
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	0	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	0	0	0	$\mathbf{e}_{41}$	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	0	0	0	$\mathbf{e}_{42}$	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	0	0	0	$\mathbf{e}_{43}$	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{1}$	$\mathbf{e}_{4}$
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_{4}$	0

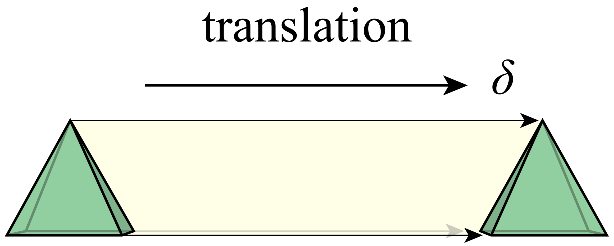
4D Geometric Antiproduct

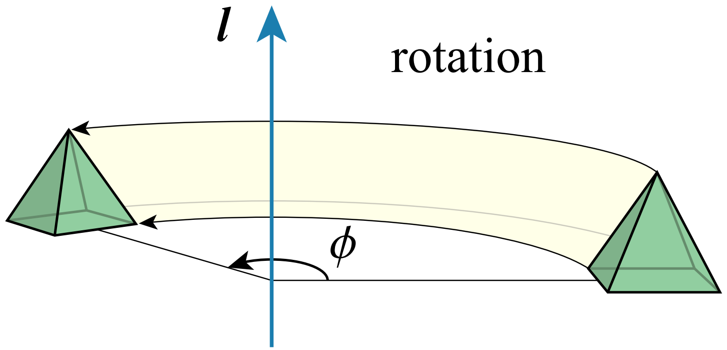
Geometric Antiproduct **a** ⟇ **b**

a \ b	1	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	𝟙
1	0	0	0	0	$\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	0	**1**
$\mathbf{e}_{1}$	0	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	0	0	0	**1**	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	0	$\mathbf{e}_{1}$
$\mathbf{e}_{2}$	0	0	0	0	$-\mathbf{e}_{31}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	0	0	0	$\mathbf{e}_{12}$	**1**	$-\mathbf{e}_{23}$	0	$\mathbf{e}_{2}$
$\mathbf{e}_{3}$	0	0	0	0	$-\mathbf{e}_{12}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	0	0	0	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	**1**	0	$\mathbf{e}_{3}$
$\mathbf{e}_{4}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	−𝟙	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{2}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	**1**	$\mathbf{e}_{4}$
$\mathbf{e}_{41}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{423}$	−𝟙	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	**−1**	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{1}$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{43}$	−𝟙	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}$	**−1**	$\mathbf{e}_{23}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423}$	$\mathbf{e}_{2}$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	$\mathbf{e}_{12}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	−𝟙	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	**−1**	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{3}$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	$\mathbf{e}_{1}$	**−1**	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	0	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	**−1**	$\mathbf{e}_{23}$	0	0	0	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	**−1**	0	0	0	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	$-\mathbf{e}_{1}$	**−1**	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	𝟙	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	$-\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	**−1**	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423}$	$\mathbf{e}_{3}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$\mathbf{e}_{43}$	𝟙	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	**−1**	$-\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	𝟙	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	**−1**	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
𝟙	**1**	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	𝟙

Proper Euclidean Isometries

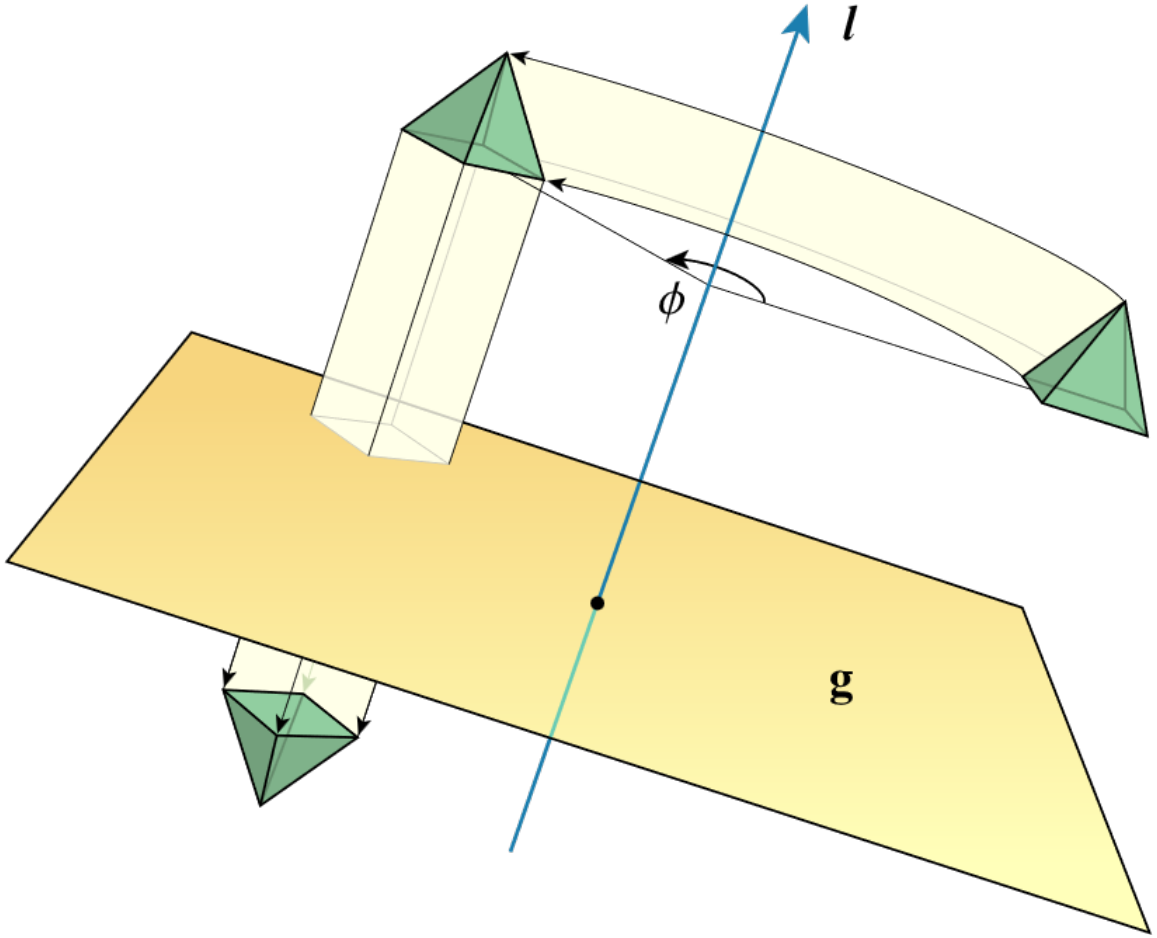




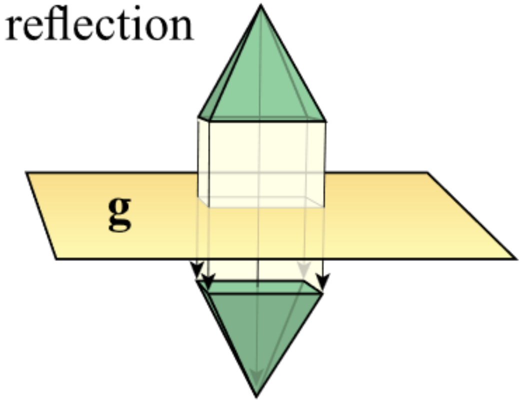


Improper Euclidean Isometries

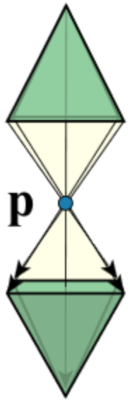
general rotoreflection



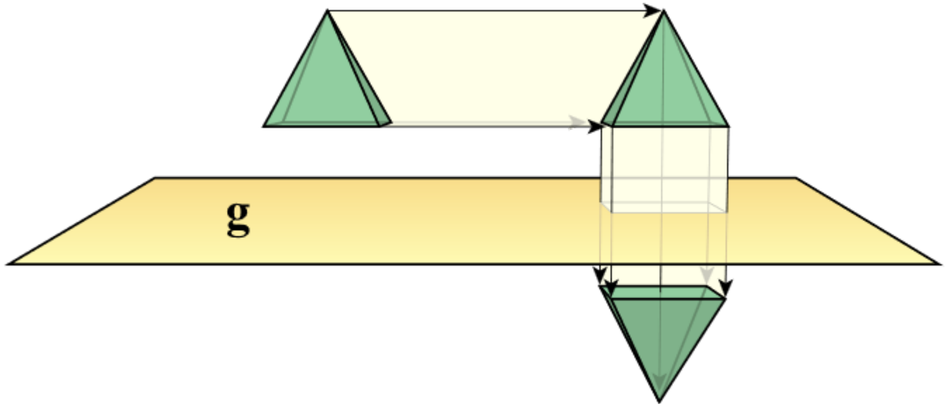
reflection



inversion



transflection



Geometric Product

- Geometric **product** in 4D space fixes the origin
- Cannot perform transformations we want

- Geometric **antiproduct** performs Euclidean isometries
- Uses sandwiching similar to quaternions

Plane Reflection

- Sandwich antiproduct with plane **g** performs reflection:

$$\mathbf{u}' = \mathbf{g} \veedot \mathbf{u} \veedot \mathbf{g}$$

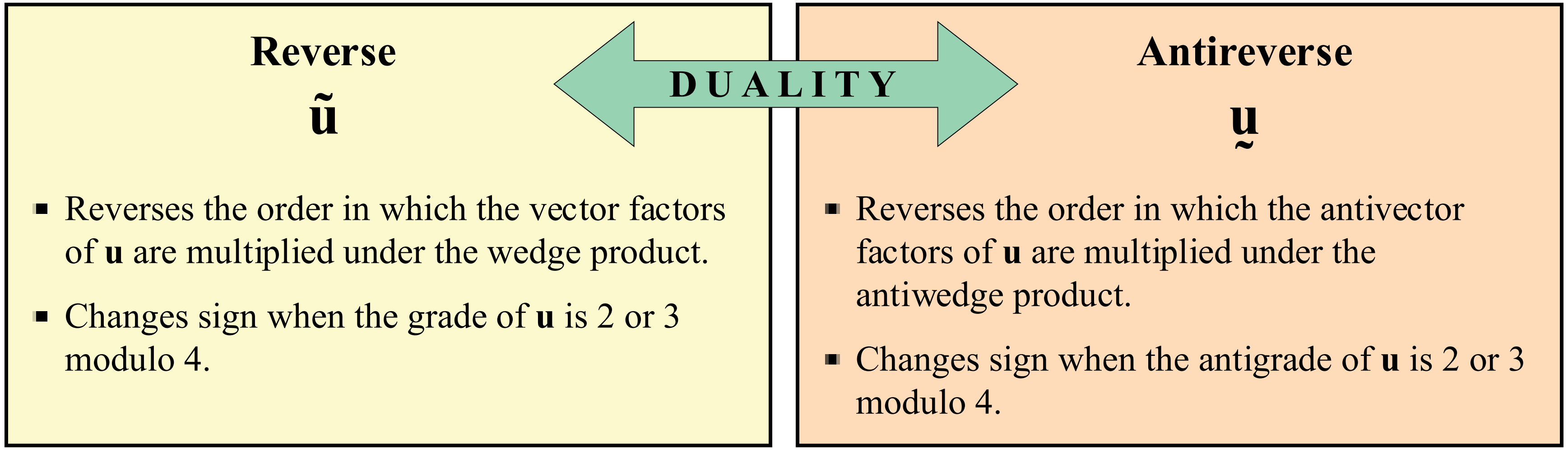
- Multiple reflections stack outward from **u**:

$$\mathbf{u}' = (\mathbf{h} \veedot \mathbf{g}) \veedot \mathbf{u} \veedot (\mathbf{g} \veedot \mathbf{h})$$

- Basis for all Euclidean isometries

Reverse and Antireverse

- Multiply vector or antivector factors in reverse order



Reverse

$\tilde{\mathbf{u}}$

- Reverses the order in which the vector factors of **u** are multiplied under the wedge product.
- Changes sign when the grade of **u** is 2 or 3 modulo 4.

DUALITY

Antireverse

$\underset{\sim}{\mathbf{u}}$

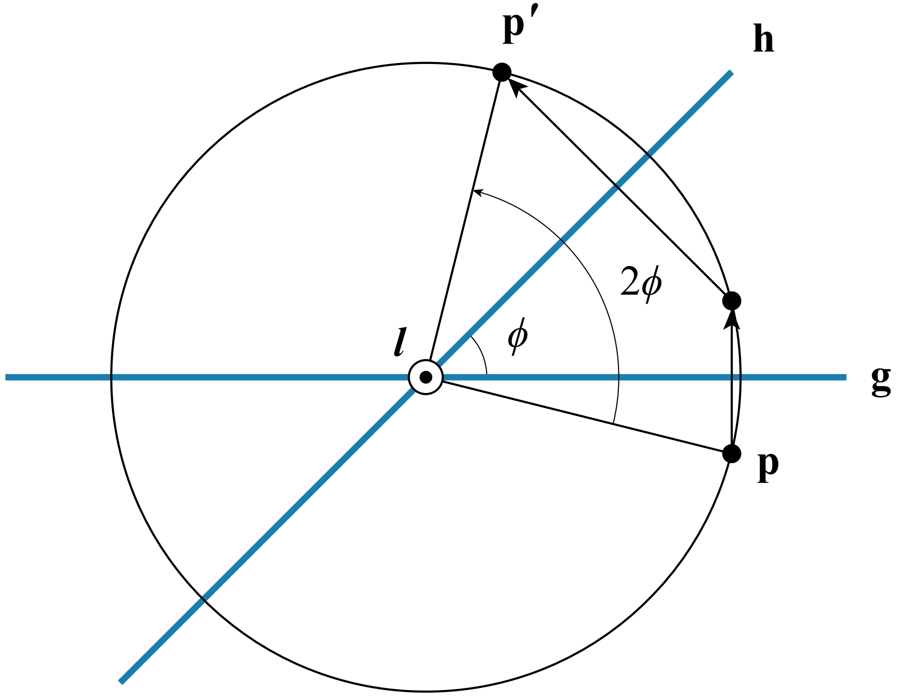
- Reverses the order in which the antivector factors of **u** are multiplied under the antiwedge product.
- Changes sign when the antigrade of **u** is 2 or 3 modulo 4.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\tilde{\mathbf{u}}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$\mathbb{1}$
$\underset{\sim}{\mathbf{u}}$	$\mathbf{1}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Rotation about a Line

- Let **g** and **h** be planes meeting at an angle ϕ
- Reflection across **g** followed by **h** is rotation through 2ϕ about line ***l*** where planes intersect

$$\boldsymbol{l} = \frac{\mathbf{h} \vee \mathbf{g}}{\left\|\mathbf{h} \vee \mathbf{g}\right\|_{\bigcirc}}$$



Rotation about a Line

- Planes multiply together under geometric antiproduct to form rotation operator **R**

$$\mathbf{p}' = \mathbf{h} \mathbin{⟇} \left(\mathbf{g} \mathbin{⟇} \mathbf{p} \mathbin{⟇} \mathbf{g}\right) \mathbin{⟇} \mathbf{h}$$

$$\mathbf{p}' = \mathbf{R} \mathbin{⟇} \mathbf{p} \mathbin{⟇} \underset{\sim}{\mathbf{R}}$$

$$\mathbf{R} = \mathbf{h} \mathbin{⟇} \mathbf{g}$$

Rotation about a Line

- General form of rotation operator **R**:

$$\mathbf{R} = \boldsymbol{l} \sin\phi + \mathbb{1} \cos\phi$$

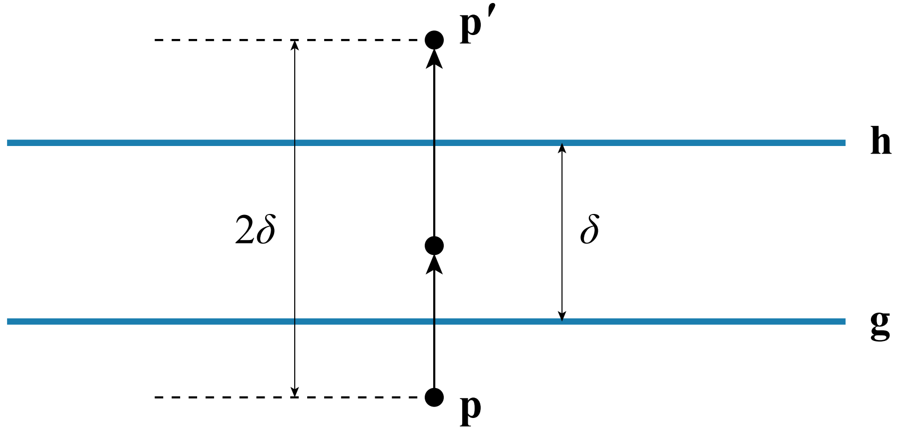
- Rotates through angle 2ϕ about unitized line ***l***

$$\mathbf{u'} = \mathbf{R} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{R}}$$

- Rotates any geometry and even other operators

Translation

- If planes **g** and **h** are parallel, result is a translation
- Translation goes along normal direction by twice the distance δ between the planes



Translation

- General form of translation operator **T**:

$$\mathbf{T} = \tau_x \mathbf{e}_{23} + \tau_y \mathbf{e}_{31} + \tau_z \mathbf{e}_{12} + \mathbb{1}$$

- Translates by displacement vector 2τ

$$\mathbf{u}' = \mathbf{T} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{T}}$$

- Translates any geometry and even other operators

Euclidean Isometry Operators

- Sandwiches with geometric antiproduct perform Euclidean isometries
- Motor = MOtion operaTOR
- Flector = reFLECtion operaTOR

Motor

- General form of a motor:

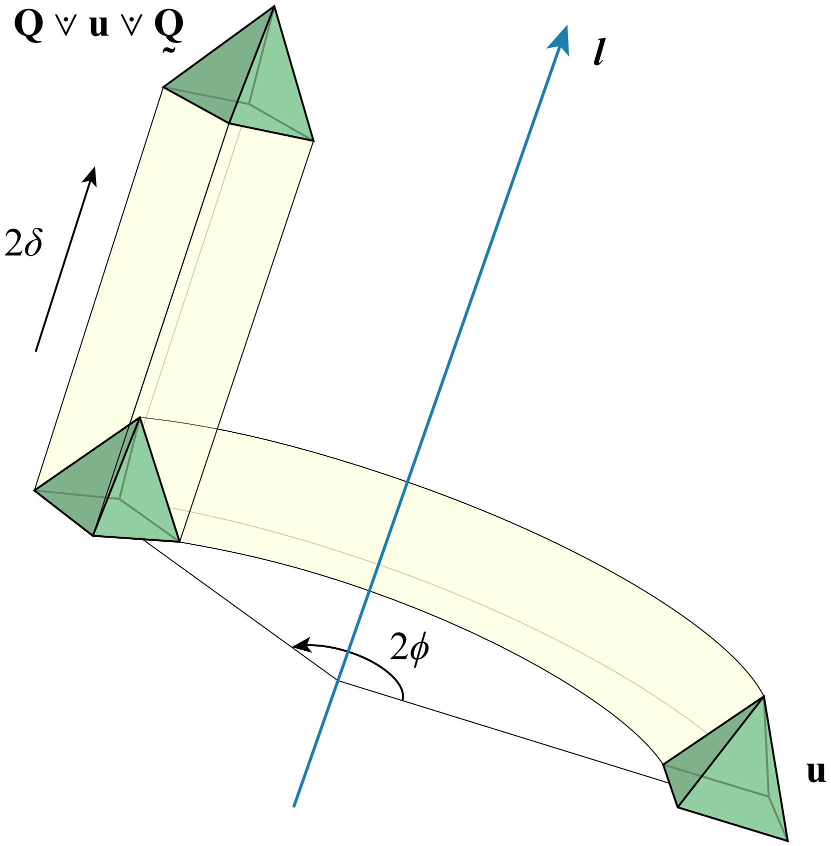
$$\mathbf{Q} = \underbrace{Q_{vx}\mathbf{e}_{41} + Q_{vy}\mathbf{e}_{42} + Q_{vz}\mathbf{e}_{43} + Q_{vw}\mathbb{1}}_{\text{Rotation Quaternion}} + \underbrace{Q_{mx}\mathbf{e}_{23} + Q_{my}\mathbf{e}_{31} + Q_{mz}\mathbf{e}_{12} + Q_{mw}\mathbf{1}}_{\text{Moment and Displacement}}$$

- Performs any combination of rotations and translations

$$\mathbf{u}' = \mathbf{Q} \mathbin{\dot{\vee}} \mathbf{u} \vee \underset{\sim}{\mathbf{Q}}$$

Sec. 3.6.1

Motor



$$\mathbf{Q} = \exp_{\dot\vee}\left[(\delta\mathbf{1} + \phi\mathbb{1}) \dot\vee \boldsymbol{l}\right] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

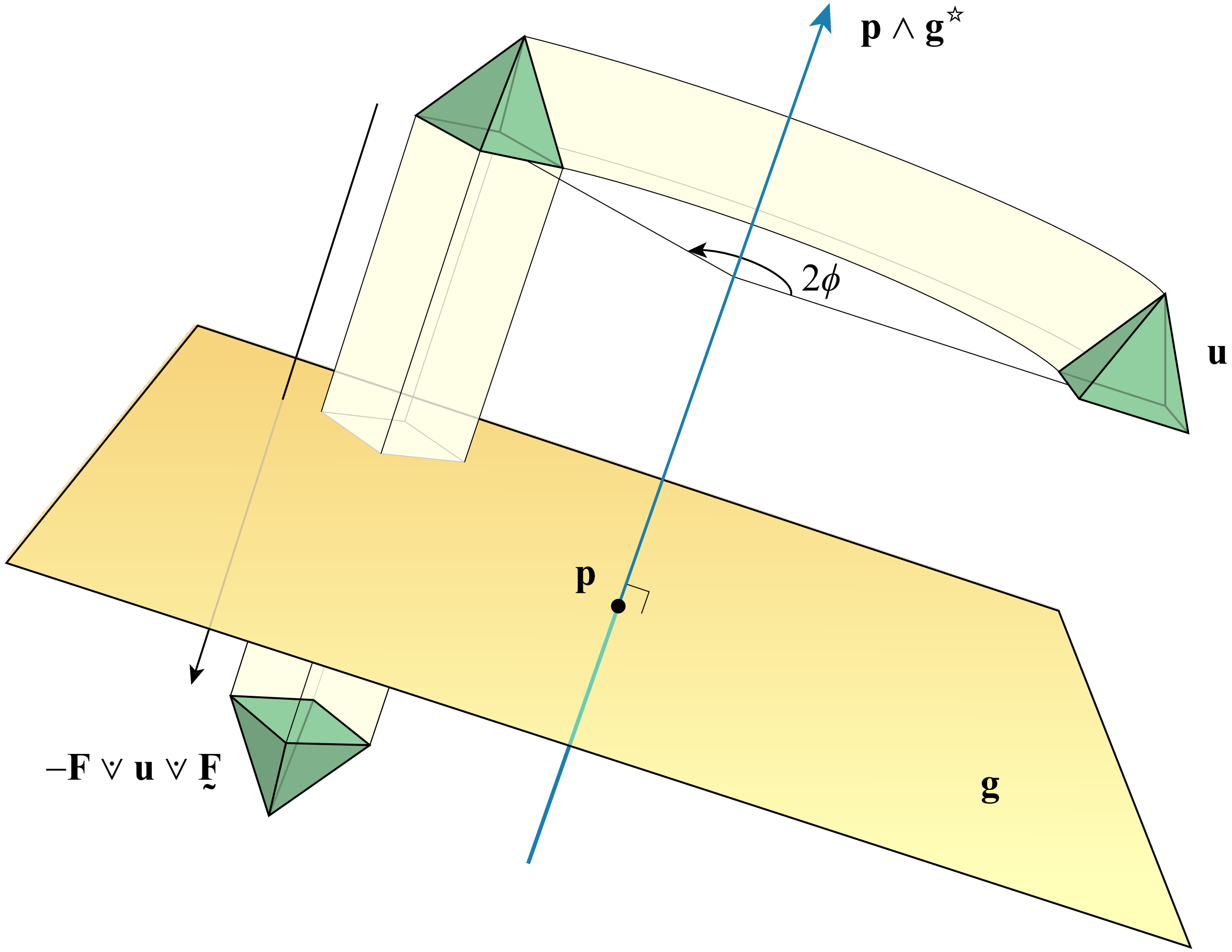
Flector

- General form of a flector:

$$\mathbf{F} = \underbrace{F_{px}\mathbf{e}_1 + F_{py}\mathbf{e}_2 + F_{pz}\mathbf{e}_3 + F_{pw}\mathbf{e}_4}_{\text{Point}} + \underbrace{F_{gx}\mathbf{e}_{423} + F_{gy}\mathbf{e}_{431} + F_{gz}\mathbf{e}_{412} + F_{gw}\mathbf{e}_{321}}_{\text{Plane}}$$

- Performs any combination of rotoreflections

Flector



$$\mathbf{F} = \mathbf{p} \sin\phi + \mathbf{g} \cos\phi$$

Motor Parameterization

- A motion operator is parameterized by:
    - A unitized line ***l***
    - A rotation angle ϕ
    - A displacement distance δ

- Exponential with respect to geometric antiproduct:

$$\mathbf{Q} = \exp_{\mathbin{\bar{\vee}}}\left[(\delta\mathbf{1} + \phi\mathbb{1}) \mathbin{\bar{\vee}} \boldsymbol{l}\right] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

- $\delta\mathbf{1} + \phi\mathbb{1}$ is *pitch* of screw transformation

Motor Parameterization

- Given arbitrary motor **Q**, can calculate parameters

$$\mathbf{Q} = Q_{vx}\mathbf{e}_{41} + Q_{vy}\mathbf{e}_{42} + Q_{vz}\mathbf{e}_{43} + Q_{vw}\mathbb{1} + Q_{mx}\mathbf{e}_{23} + Q_{my}\mathbf{e}_{31} + Q_{mz}\mathbf{e}_{12} + Q_{mw}\mathbf{1}$$

$$\mathbf{Q} = \exp_{\dot{\vee}}\left[(\delta\mathbf{1} + \phi\mathbb{1}) \mathbin{\dot{\vee}} \boldsymbol{l}\right] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

$$s = \sin\phi = \sqrt{1 - Q_{vw}^{2}} \qquad \delta = -\frac{Q_{mw}}{s} \qquad \phi = \tan^{-1}\left(\frac{s}{Q_{vw}}\right)$$

$$\boldsymbol{l}_{\mathbf{v}} = \frac{1}{s}\mathbf{Q}_{vxyz} \qquad \boldsymbol{l}_{\mathbf{m}} = \frac{1}{s}\left(\mathbf{Q}_{mxyz} + \frac{Q_{vw}Q_{mw}}{s^{2}}\mathbf{Q}_{vxyz}\right)$$

Operator Norms

- Geometric norm of operator is half the distance origin is moved by the operator

$$\|\mathbf{Q}\| = \mathbf{1}\sqrt{Q_{mx}^2 + Q_{my}^2 + Q_{mz}^2 + Q_{mw}^2} + \mathbb{1}\sqrt{Q_{vx}^2 + Q_{vy}^2 + Q_{vz}^2 + Q_{vw}^2}$$

Motor Interpolation

- To interpolate from motor $\mathbf{Q}_1$ to motor $\mathbf{Q}_2$, first calculate

$$\mathbf{Q}_0 = \mathbf{Q}_2 \mathbin{\dot{\vee}} \mathbf{Q}_1^{-1} = \mathbf{Q}_2 \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{Q}}_1$$

- Then calculate parameters ***l***, δ, and ϕ for $\mathbf{Q}_0$

- Interpolate from identity $\mathbb{1}$ to $\mathbf{Q}_0$ with

$$\mathbf{Q}(t) = \exp_{\dot{\vee}}\left[t(\delta\mathbf{1} + \phi\mathbb{1}) \mathbin{\dot{\vee}} \boldsymbol{l}\right] = \boldsymbol{l}\sin(t\phi) - \boldsymbol{l}^{\star} t\delta\cos(t\phi) - t\delta\sin(t\phi) + \mathbb{1}\cos(t\phi)$$

- Finally, calculate $\mathbf{Q}(t) \mathbin{\dot{\vee}} \mathbf{Q}_1$

Motor Interpolation

- That can be computationally expensive

- Approximate interpolation is often acceptable:

$$\mathbf{Q}(t) = (1 - t)\mathbf{Q}_1 + t\mathbf{Q}_2$$

- This needs to be unitized and constrained

$$\frac{\mathbf{Q}}{\left\|\mathbf{Q}_\mathbf{v}\right\|} \dot{\vee} \left(-\frac{\mathbf{Q}_\mathbf{v} \cdot \mathbf{Q}_\mathbf{m}}{\mathbf{Q}_\mathbf{v}^2} \mathbf{1} + \mathbb{1} \right) = \frac{1}{\left\|\mathbf{Q}_\mathbf{v}\right\|} \left[\mathbf{Q} - \frac{\mathbf{Q}_\mathbf{v} \cdot \mathbf{Q}_\mathbf{m}}{\mathbf{Q}_\mathbf{v}^2} \left(Q_{vx}\mathbf{e}_{23} + Q_{vy}\mathbf{e}_{31} + Q_{vz}\mathbf{e}_{12} + Q_{vw} \right) \right]$$

Square Root of Motor

- Special case of interpolation from $\mathbb{1}$ to $\mathbf{Q}$ when $t = 1/2$

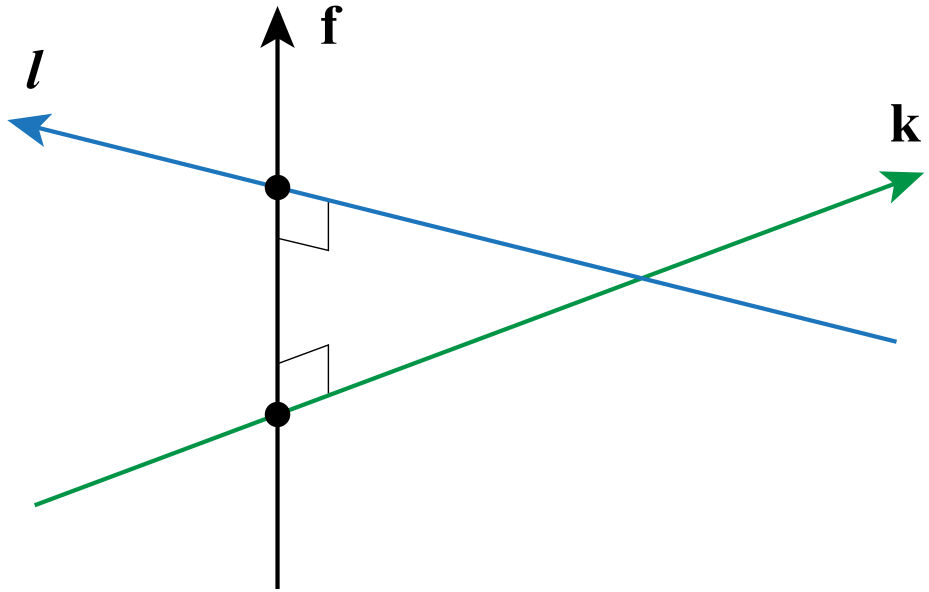
$$\sqrt[\veedot]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\sqrt{2 + 2Q_{\mathbb{1}}}} \veedot \left(\mathbb{1} - \frac{\mathbf{Q}_{\mathbf{1}}}{2 + 2Q_{\mathbb{1}}} \mathbf{1} \right)$$

- For simple motor (pure rotation or translation), this simplifies:

$$\sqrt[\veedot]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\left\| \mathbf{Q} + \mathbb{1} \right\|_{\bigcirc}}$$

Line to Line Motion

- Let **k** and ***l*** be lines separated by distance δ with angle ϕ between directions
- Operator $\boldsymbol{l} \mathbin{\dot\vee} \underset{\sim}{\mathbf{k}}$ rotates by 2ϕ and translates by distance 2δ about line **f** connecting closest points
- Square root of this operator transforms line **k** into line ***l***



Motor-Point Transformation

- 25 multiply-adds:

$$\mathbf{p}'_{xyz} = \mathbf{p}_{xyz} + 2(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a} - Q_{mw}p_w\mathbf{v})$$
$$p'_w = p_w$$

$$\mathbf{a} = \mathbf{v} \times \mathbf{p}_{xyz} + p_w\mathbf{m}$$

$$\mathbf{v} = (Q_{vx}, Q_{vy}, Q_{vz})$$
$$\mathbf{m} = (Q_{mx}, Q_{my}, Q_{mz})$$

- 3×4 matrix transformation only requires 12 multiply-adds, (or just 9 if $p_w = 1$)

Motor-Line Transformation

- 54 multiply-adds:

$$\boldsymbol{l}'_{\mathbf{v}} = \boldsymbol{l}_{\mathbf{v}} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a}\right)$$

$$\boldsymbol{l}'_{\mathbf{m}} = \boldsymbol{l}_{\mathbf{m}} + 2\left[Q_{mw}\mathbf{a} + Q_{vw}(\mathbf{b} + \mathbf{c}) + \mathbf{v} \times (\mathbf{b} + \mathbf{c}) + \mathbf{m} \times \mathbf{a}\right]$$

$$\mathbf{a} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{v}} \qquad \mathbf{b} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{m}} \qquad \mathbf{c} = \mathbf{m} \times \boldsymbol{l}_{\mathbf{v}}$$

- 6×6 matrix transformation only requires 27 multiply-adds

Motor-Plane Transformation

- 35 multiply-adds:

$$\mathbf{g}'_{xyz} = \mathbf{g}_{xyz} + 2(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a})$$

$$g'_w = g_w + 2\left[\left(\mathbf{m} \times \mathbf{g}_{xyz} + Q_{mw}\mathbf{g}_{xyz}\right) \cdot \mathbf{v} - Q_{vw}\left(\mathbf{m} \cdot \mathbf{g}_{xyz}\right)\right]$$

$$\mathbf{a} = \mathbf{v} \times \mathbf{g}_{xyz}$$

- 4×4 matrix transformation only requires 13 multiply-adds

Motor to Matrix

$$\mathbf{A_Q} = \begin{bmatrix} 1-2\left(Q_{vy}^2+Q_{vz}^2\right) & 2Q_{vx}Q_{vy} & 2Q_{vz}Q_{vx} & 2\left(Q_{vy}Q_{mz}-Q_{vz}Q_{my}\right) \\ 2Q_{vx}Q_{vy} & 1-2\left(Q_{vz}^2+Q_{vx}^2\right) & 2Q_{vy}Q_{vz} & 2\left(Q_{vz}Q_{mx}-Q_{vx}Q_{mz}\right) \\ 2Q_{vz}Q_{vx} & 2Q_{vy}Q_{vz} & 1-2\left(Q_{vx}^2+Q_{vy}^2\right) & 2\left(Q_{vx}Q_{my}-Q_{vy}Q_{mx}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B_Q} = \begin{bmatrix} 0 & -2Q_{vz}Q_{vw} & 2Q_{vy}Q_{vw} & 2\left(Q_{vw}Q_{mx}-Q_{vx}Q_{mw}\right) \\ 2Q_{vz}Q_{vw} & 0 & -2Q_{vx}Q_{vw} & 2\left(Q_{vw}Q_{my}-Q_{vy}Q_{mw}\right) \\ -2Q_{vy}Q_{vw} & 2Q_{vx}Q_{vw} & 0 & 2\left(Q_{vw}Q_{mz}-Q_{vz}Q_{mw}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M_Q} = \mathbf{A_Q} + \mathbf{B_Q} \qquad \mathbf{M_Q^{-1}} = \mathbf{A_Q} - \mathbf{B_Q}$$

Motor Composition

- 48 multiply-adds:

$$
\begin{aligned}
\mathbf{Q} \mathbin{\dot{\vee}} \mathbf{R} = {} & \left(Q_{vw}R_{vx} + Q_{vx}R_{vw} + Q_{vy}R_{vz} - Q_{vz}R_{vy}\right)\mathbf{e}_{41} \\
& + \left(Q_{vw}R_{vy} - Q_{vx}R_{vz} + Q_{vy}R_{vw} + Q_{vz}R_{vx}\right)\mathbf{e}_{42} \\
& + \left(Q_{vw}R_{vz} + Q_{vx}R_{vy} - Q_{vy}R_{vx} + Q_{vz}R_{vw}\right)\mathbf{e}_{43} \\
& + \left(Q_{vw}R_{vw} - Q_{vx}R_{vx} - Q_{vy}R_{vy} - Q_{vz}R_{vz}\right)\mathbb{1} \\
& + \left(Q_{mw}R_{vx} + Q_{mx}R_{vw} + Q_{my}R_{vz} - Q_{mz}R_{vy} + Q_{vw}R_{mx} + Q_{vx}R_{mw} + Q_{vy}R_{mz} - Q_{vz}R_{my}\right)\mathbf{e}_{23} \\
& + \left(Q_{mw}R_{vy} - Q_{mx}R_{vz} + Q_{my}R_{vw} + Q_{mz}R_{vx} + Q_{vw}R_{my} - Q_{vx}R_{mz} + Q_{vy}R_{mw} + Q_{vz}R_{mx}\right)\mathbf{e}_{31} \\
& + \left(Q_{mw}R_{vz} + Q_{mx}R_{vy} - Q_{my}R_{vx} + Q_{mz}R_{vw} + Q_{vw}R_{mz} + Q_{vx}R_{my} - Q_{vy}R_{mx} + Q_{vz}R_{mw}\right)\mathbf{e}_{12} \\
& + \left(Q_{mw}R_{vw} - Q_{mx}R_{vx} - Q_{my}R_{vy} - Q_{mz}R_{vz} + Q_{vw}R_{mw} - Q_{vx}R_{mx} - Q_{vy}R_{my} + Q_{vz}R_{mz}\right)\mathbf{1}
\end{aligned}
$$

- Composition of equivalent 3×4 matrices requires 33 multiply-adds

Motor and Matrix

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_x \\ m_{21} & m_{22} & m_{23} & t_y \\ m_{31} & m_{32} & m_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Q} = Q_{vx}\mathbf{e}_{41} + Q_{vy}\mathbf{e}_{42} + Q_{vz}\mathbf{e}_{43} + Q_{vw}\mathbb{1} + Q_{mx}\mathbf{e}_{23} + Q_{my}\mathbf{e}_{31} + Q_{mz}\mathbf{e}_{12} + Q_{mw}\mathbf{1}$$

$$\mathbf{M} = \begin{bmatrix} 1-2\left(Q_{vy}^2+Q_{vz}^2\right) & 2\left(Q_{vx}Q_{vy}-Q_{vz}Q_{vw}\right) & 2\left(Q_{vz}Q_{vx}+Q_{vy}Q_{vw}\right) & 2\left(Q_{vy}Q_{mz}-Q_{vz}Q_{my}+Q_{vw}Q_{mx}-Q_{vx}Q_{mw}\right) \\ 2\left(Q_{vx}Q_{vy}+Q_{vz}Q_{vw}\right) & 1-2\left(Q_{vz}^2+Q_{vx}^2\right) & 2\left(Q_{vy}Q_{vz}-Q_{vx}Q_{vw}\right) & 2\left(Q_{vz}Q_{mx}-Q_{vx}Q_{mz}+Q_{vw}Q_{my}-Q_{vy}Q_{mw}\right) \\ 2\left(Q_{vz}Q_{vx}-Q_{vy}Q_{vw}\right) & 2\left(Q_{vy}Q_{vz}+Q_{vx}Q_{vw}\right) & 1-2\left(Q_{vx}^2+Q_{vy}^2\right) & 2\left(Q_{vx}Q_{my}-Q_{vy}Q_{mx}+Q_{vw}Q_{mz}-Q_{vz}Q_{mw}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

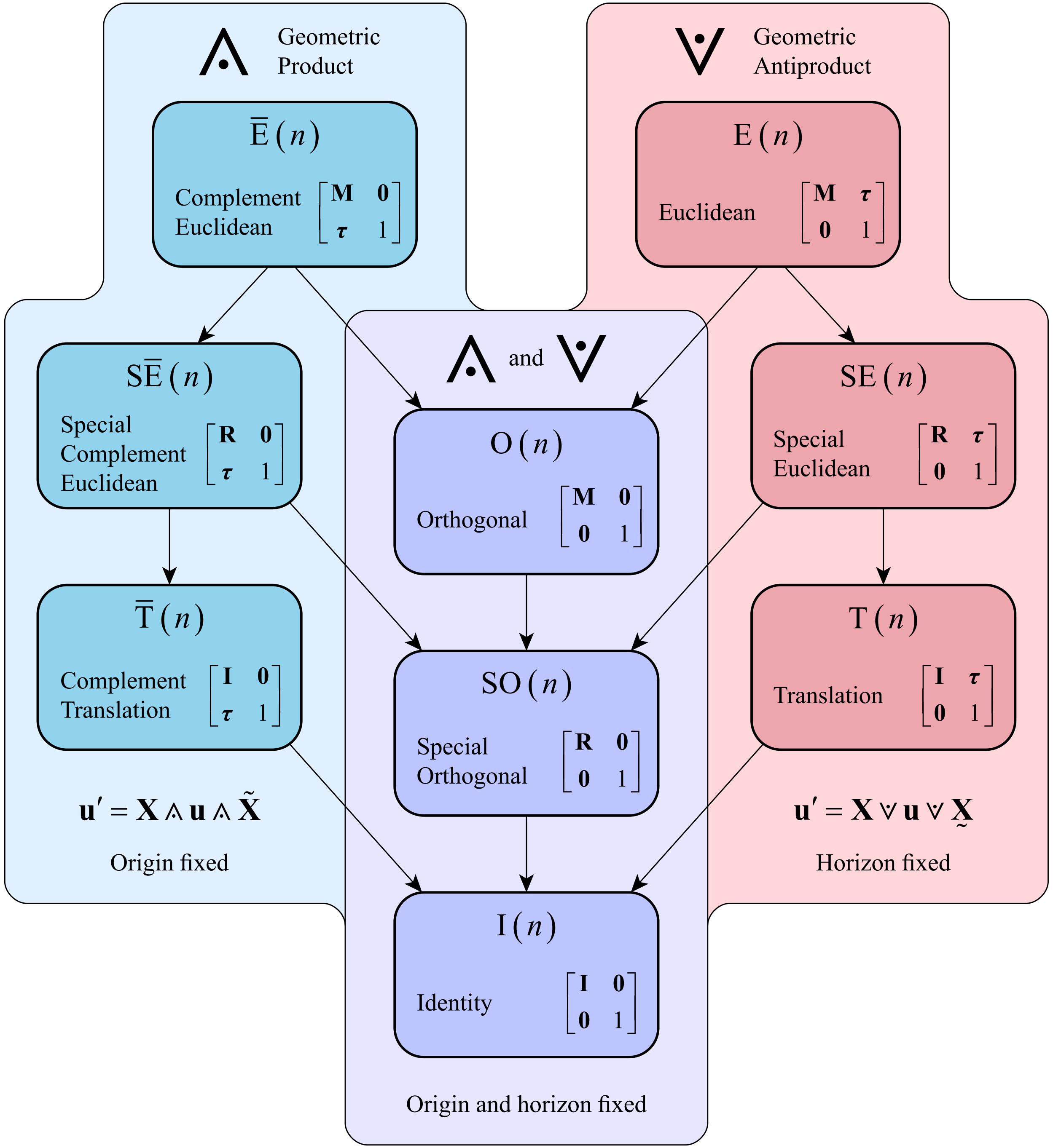
Matrix Advantages

- Can represent more transformations
- Can read off origin and axis directions in transformed space
- Faster to transform objects
- Faster to compose

Motor Advantages

- Smaller storage requirements
  - Usually 8 floats, but can reduce to 6
- Inversion is trivial
  - Just reverse, negating six bivector components
- Better parameterization
- Better interpolation properties

Transformation Groups



⟑ Geometric Product

$\overline{\mathrm{E}}(n)$

Complement Euclidean $\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \boldsymbol{\tau} & 1 \end{bmatrix}$

$\mathrm{S}\overline{\mathrm{E}}(n)$

Special Complement Euclidean $\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \boldsymbol{\tau} & 1 \end{bmatrix}$

$\overline{\mathrm{T}}(n)$

Complement Translation $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \boldsymbol{\tau} & 1 \end{bmatrix}$

$$\mathbf{u}' = \mathbf{X} ⟑ \mathbf{u} ⟑ \tilde{\mathbf{X}}$$

Origin fixed

⟑ and ⟇

$\mathrm{O}(n)$

Orthogonal $\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathrm{SO}(n)$

Special Orthogonal $\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathrm{I}(n)$

Identity $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$

Origin and horizon fixed

⟇ Geometric Antiproduct

$\mathrm{E}(n)$

Euclidean $\begin{bmatrix} \mathbf{M} & \boldsymbol{\tau} \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathrm{SE}(n)$

Special Euclidean $\begin{bmatrix} \mathbf{R} & \boldsymbol{\tau} \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathrm{T}(n)$

Translation $\begin{bmatrix} \mathbf{I} & \boldsymbol{\tau} \\ \mathbf{0} & 1 \end{bmatrix}$

$$\mathbf{u}' = \mathbf{X} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{X}}$$

Horizon fixed

Transwedge Product

- Generalization of exterior and interior products

$$\mathbf{a} \underset{k}{\Wedge} \mathbf{b} = \sum_{c \in \mathcal{B}_k} \left(\underline{\mathbf{c}} \vee \mathbf{a} \right) \wedge \left(\mathbf{b} \vee \mathbf{c}^{\star} \right)$$

$$\mathbf{a} \underset{k}{\Wedge} \mathbf{b} = \sum_{c \in \mathcal{B}_k} \left(\mathbf{c}_{\star} \vee \mathbf{a} \right) \wedge \left(\mathbf{b} \vee \overline{\mathbf{c}} \right)$$

$$\mathcal{B}_k = \text{set of basis elements of grade } k$$

Transwedge Product

- Order 0 transwedge product is the exterior product:

$$\mathbf{a} \underset{0}{\owedge} \mathbf{b} = \mathbf{a} \wedge \mathbf{b}$$

- Order *m* transwedge product is the interior product, where *m* is minimum grade of operands

$$\mathbf{a} \underset{\min(\operatorname{gr}(\mathbf{a}), \operatorname{gr}(\mathbf{b}))}{\owedge} \mathbf{b} = \begin{cases} \mathbf{b} \vee \mathbf{a}^{\bigstar}, & \text{if } \operatorname{gr}(\mathbf{a}) \leq \operatorname{gr}(\mathbf{b}); \\ \mathbf{b}_{\bigstar} \vee \mathbf{a}, & \text{if } \operatorname{gr}(\mathbf{a}) \geq \operatorname{gr}(\mathbf{b}). \end{cases}$$

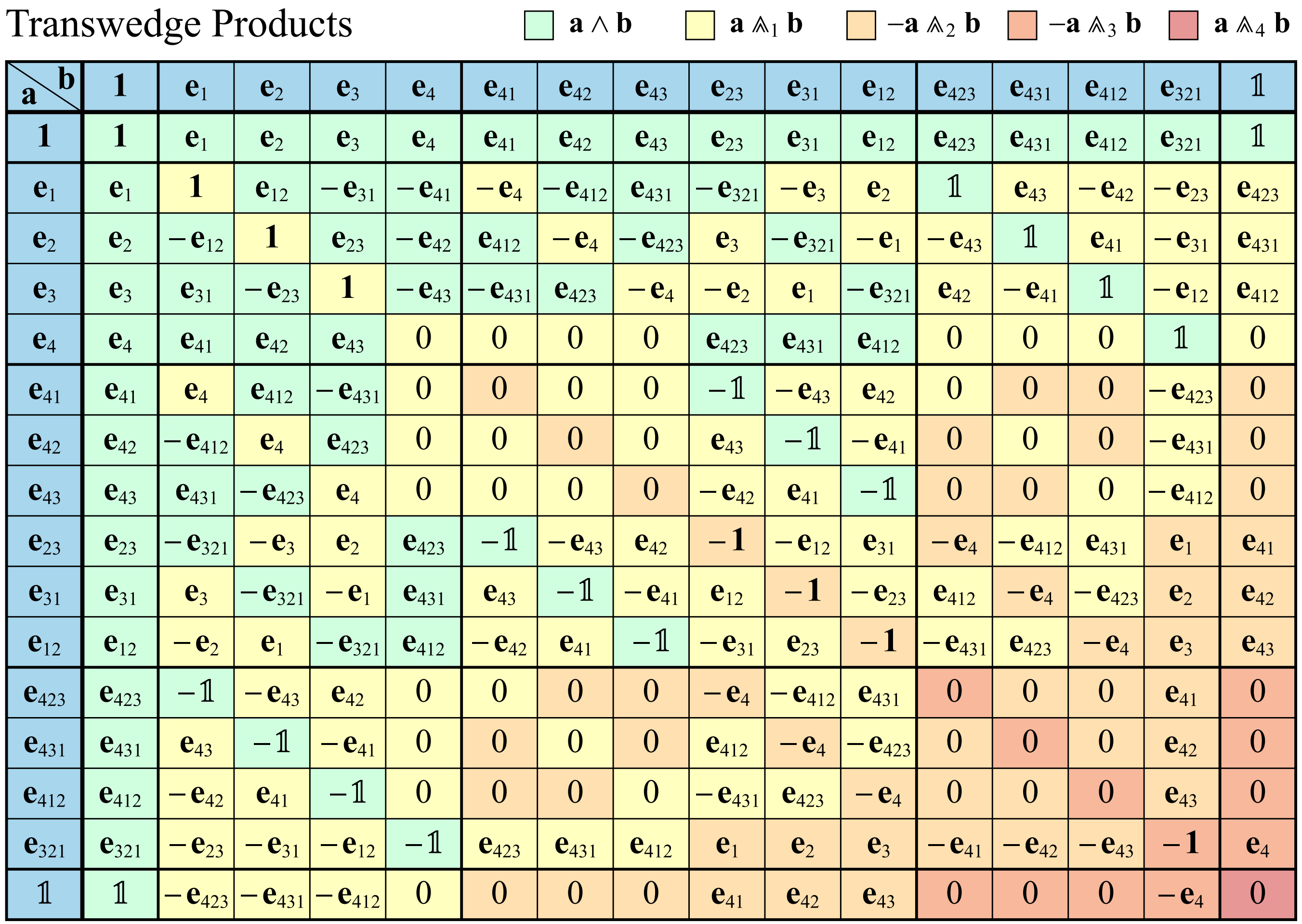
Transwedge Product

- Orders in between 0 and *m*, exclusive, are “liminal” products
    - Neither exterior nor interior

- Geometric product decomposes into transwedge products as

$$\mathbf{a} \wedgedot \mathbf{b} = \sum_{k=0}^{m} (-1)^{k(k-1)/2} \, \mathbf{a} \underset{k}{\mathbin{\overset{\wedge}{\wedge}}} \mathbf{b}$$

Transwedge Product



Transwedge Products

🟩 $\mathbf{a} \wedge \mathbf{b}$ 🟨 $\mathbf{a} ⩓_{1} \mathbf{b}$ 🟧 $-\mathbf{a} ⩓_{2} \mathbf{b}$ 🟧 $-\mathbf{a} ⩓_{3} \mathbf{b}$ 🟥 $\mathbf{a} ⩓_{4} \mathbf{b}$

$\mathbf{b}$ $\mathbf{a}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_{1}$	$\mathbf{e}_{1}$	$\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{2}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	$\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{43}$	$\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{3}$	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{1}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbb{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{4}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	$-\mathbf{e}_{423}$	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{4}$	$\mathbf{e}_{423}$	0	0	0	0	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	$-\mathbf{e}_{431}$	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{4}$	0	0	0	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	$-\mathbf{e}_{412}$	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$-\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_{1}$	$\mathbf{e}_{41}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	$-\mathbf{1}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{2}$	$\mathbf{e}_{42}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{3}$	$\mathbf{e}_{43}$
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	0	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	0	0	0	$\mathbf{e}_{41}$	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	0	0	0	$\mathbf{e}_{42}$	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	0	0	0	$\mathbf{e}_{43}$	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{1}$	$\mathbf{e}_{4}$
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_{4}$	0

Transwedge Product

- In 4D algebra, we previously wrote

$$\mathbf{A} \mathbin{\dot{\wedge}} \mathbf{B} = \mathbf{A} \wedge \mathbf{B} + \mathbf{A} \times \mathbf{B} + \mathbf{A} \bullet \tilde{\mathbf{B}}$$

- Now, we have something better:

$$\mathbf{A} \mathbin{\dot{\wedge}} \mathbf{B} = \mathbf{A} \underset{0}{\Wedge} \mathbf{B} + \mathbf{A} \underset{1}{\Wedge} \mathbf{B} - \mathbf{A} \underset{2}{\Wedge} \mathbf{B}$$

$$= \mathbf{A} \wedge \mathbf{B} + \mathbf{A} \underset{1}{\Wedge} \mathbf{B} + \mathbf{A} \bullet \tilde{\mathbf{B}}$$

Transwedge Product

- This means that the geometric product is just another operation in the exterior algebra

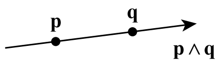
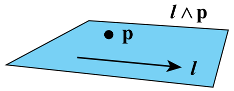
- Every product / antiproduct in the exterior algebra, including the geometric product, can be derived from 3 primitive operations:

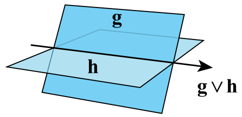
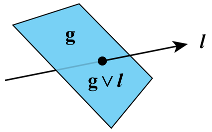
  - The wedge product $\wedge$
  - Taking of a complement $\overline{\mathbf{a}}$
  - Application of the metric $\mathbf{Ga}$

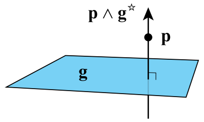
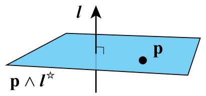
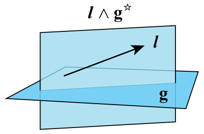
Transwedge Product

- Liminal products have geometric significance

- In 4D rigid algebra, one geometric combo conspicuously missing from our tables

Join Operation	**Illustration**
Line containing points **p** and **q**. $\mathbf{p} \wedge \mathbf{q} = (p_w q_x - p_x q_w)\mathbf{e}_{41} + (p_y q_z - p_z q_y)\mathbf{e}_{23}$ $+ (p_w q_y - p_y q_w)\mathbf{e}_{42} + (p_z q_x - p_x q_z)\mathbf{e}_{31}$ $+ (p_w q_z - p_z q_w)\mathbf{e}_{43} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$	
Plane containing line ***l*** and point **p**. $\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx} p_w)\mathbf{e}_{423}$ $+ (l_{vz} p_x - l_{vx} p_z + l_{my} p_w)\mathbf{e}_{431}$ $+ (l_{vx} p_y - l_{vy} p_x + l_{mz} p_w)\mathbf{e}_{412}$ $- (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\mathbf{e}_{321}$	

Meet Operation	**Illustration**
Line where planes **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\mathbf{e}_{41} + (g_x h_w - g_w h_x)\mathbf{e}_{23}$ $+ (g_x h_z - g_z h_x)\mathbf{e}_{42} + (g_y h_w - g_w h_y)\mathbf{e}_{31}$ $+ (g_y h_x - g_x h_y)\mathbf{e}_{43} + (g_z h_w - g_w h_z)\mathbf{e}_{12}$	
Point where plane **g** and line ***l*** intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\mathbf{e}_1$ $+ (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\mathbf{e}_2$ $+ (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\mathbf{e}_3$ $- (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\mathbf{e}_4$	

Expansion Operation	**Illustration**
Line containing point **p** and orthogonal to plane **g**. $\mathbf{p} \wedge \mathbf{g}^{\star} = -p_w g_x \mathbf{e}_{41} + (p_z g_y - p_y g_z)\mathbf{e}_{23}$ $- p_w g_y \mathbf{e}_{42} + (p_x g_z - p_z g_x)\mathbf{e}_{31}$ $- p_w g_z \mathbf{e}_{43} + (p_y g_x - p_x g_y)\mathbf{e}_{12}$	
Plane containing point **p** and orthogonal to line ***l***. $\mathbf{p} \wedge \boldsymbol{l}^{\star} = -p_w l_{vx} \mathbf{e}_{423} - p_w l_{vy} \mathbf{e}_{431} - p_w l_{vz} \mathbf{e}_{412}$ $+ (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\mathbf{e}_{321}$	
Plane containing line ***l*** and orthogonal to plane **g**. $\boldsymbol{l} \wedge \mathbf{g}^{\star} = (l_{vy} g_z - l_{vz} g_y)\mathbf{e}_{423}$ $+ (l_{vz} g_x - l_{vx} g_z)\mathbf{e}_{431}$ $+ (l_{vx} g_y - l_{vy} g_x)\mathbf{e}_{412}$ $- (l_{mx} g_x + l_{my} g_y + l_{mz} g_z)\mathbf{e}_{321}$	

Transwedge Product

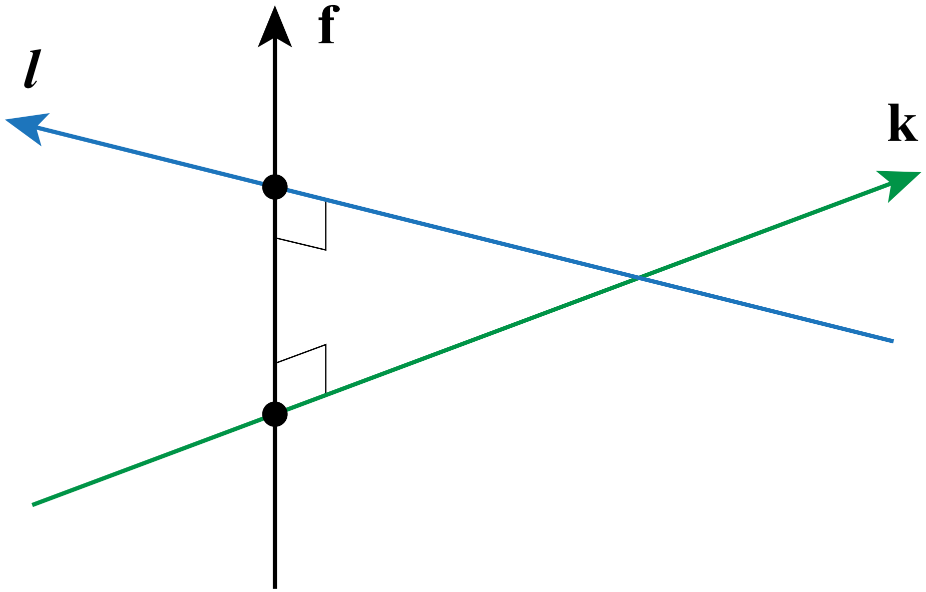
- Calculate line **f** orthogonal to two skew lines ***l*** and **k**

$$\mathbf{f} = \boldsymbol{l} \underset{1}{\veebar} \mathbf{k}$$

- Uses transwedge *antiproduct*

- Needs to be reconstrained so that

$$\mathbf{f_v} \cdot \mathbf{f_m} = 0$$



Transwedge Product

- **f** will generally need to be “constrained” so that $\mathbf{f}_{\mathbf{v}} \cdot \mathbf{f}_{\mathbf{m}} = 0$
- This can be done by dividing by dual number $\sqrt[\dot{\vee}]{\mathbf{f} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{f}}}$
- Equivalent to Gram-Schmidt orthonormalization
- This dual number norm arises from isomorphism

$$\mathbb{R}^{+}(3,0,1) \cong \mathbb{D}^{+}(3,0,0)$$

Spacetime Geometric Algebra

- Add time dimension $\mathbf{e}_0$ that squares to −1

$$\mathfrak{g} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_0 \cdot \mathbf{e}_0 = -1$$
$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$
$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$
$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$
$$\mathbf{e}_4 \cdot \mathbf{e}_4 = 0$$

Spacetime Geometric Algebra

- Position: $\mathbf{r} = ct\,\mathbf{e}_0 + x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3 + \mathbf{e}_4$
- Velocity: $\mathbf{u} = \dfrac{d\mathbf{r}}{d\tau} = \gamma c\,\mathbf{e}_0 + \gamma\dot{x}\,\mathbf{e}_1 + \gamma\dot{y}\,\mathbf{e}_2 + \gamma\dot{z}\,\mathbf{e}_3$

$$\gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - u^2/c^2}}$$

Relativistic Quaternions

- Operators in 4D algebra transfer to 5D algebra through multiplication by $\mathbf{e}_0$

- Quaternion rotation: $\mathbf{q} = q_x \mathbf{e}_{410} + q_y \mathbf{e}_{420} + q_z \mathbf{e}_{430} + q_w \mathbb{1}$ $\qquad \mathbb{1} = \mathbf{e}_{01234}$

$$\mathbf{q}(\tau) = \left(a_x \mathbf{e}_{410} + a_y \mathbf{e}_{420} + a_z \mathbf{e}_{430}\right) \sin\left(\frac{1}{2}\gamma\tau\dot{\phi}\right) + \mathbb{1} \cos\left(\frac{1}{2}\gamma\tau\dot{\phi}\right) \qquad \dot{\phi} = d\phi/dt$$

$$\mathbf{q}(\tau) = \exp_{\vee}\left(\tfrac{1}{2}\gamma\tau\dot{\phi}\boldsymbol{l}\right) \qquad \boldsymbol{l} = a_x \mathbf{e}_{410} + a_y \mathbf{e}_{420} + a_z \mathbf{e}_{430}$$

Relativistic Quaternions

- Sandwich product $\mathbf{q}(\tau) \dot{\vee} \mathbf{r} \dot{\vee} \underset{\sim}{\mathbf{q}}(\tau)$ performs an instantaneous rotation

- Any physically meaningful motion must happen over some amount of time

- Also must respect speed of light

Relativistic Quaternions

- General translation operator:

$$\mathbf{T}(\tau)=\frac{1}{2}\gamma\tau\left(-c\,\mathbf{e}_{321}+\dot{x}\,\mathbf{e}_{230}+\dot{y}\,\mathbf{e}_{310}+\dot{z}\,\mathbf{e}_{120}\right)+\mathbb{1}$$

- Strictly temporal translation:

$$\mathbf{S}(\tau)=\mathbb{1}-\frac{1}{2}\gamma c\tau\,\mathbf{e}_{321}$$

Relativistic Quaternions

- Combine rotation and temporal translation:

$$\mathbf{Q}(\tau) = \mathbf{q}(\tau) \dot{\vee} \left(\mathbb{1} - \frac{1}{2} \gamma c \tau \, \mathbf{e}_{321} \right)$$

$$\begin{aligned} \mathbf{Q}(\tau) = {} & \left(a_x \mathbf{e}_{410} + a_y \mathbf{e}_{420} + a_z \mathbf{e}_{430} \right) \sin\left(\frac{1}{2} \gamma \tau \dot{\phi} \right) + \mathbb{1} \cos\left(\frac{1}{2} \gamma \tau \dot{\phi} \right) \\ & - \frac{1}{2} \gamma c \tau \left(a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3 \right) \sin\left(\frac{1}{2} \gamma \tau \dot{\phi} \right) - \frac{1}{2} \gamma c \tau \, \mathbf{e}_{321} \cos\left(\frac{1}{2} \gamma \tau \dot{\phi} \right) \end{aligned}$$

Relativistic Quaternions

- Set $\tau = t/\gamma$

- Then $\mathbf{q}(\tau) ⟇ \mathbf{r} ⟇ \utilde{\mathbf{q}}(\tau)$ rotates **r** through angle $\dot{\phi}t$ and adds ct to the time coordinate

- Generically, *s* coordinates are *q* coordinates times $-\frac{1}{2}\gamma c\tau$

$$\mathbf{Q} = q_x\mathbf{e}_{410} + q_y\mathbf{e}_{420} + q_z\mathbf{e}_{430} + q_w\mathbb{1} + s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + s_w\mathbf{e}_{321}$$

Relativistic Quaternions

- Operator **Q** is equivalent to 5 x 5 matrix

$$\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & 0 & -2\left(q_x s_x + q_y s_y + q_z s_z + q_w s_w\right) \\ 0 & 1 - 2q_y^2 - 2q_z^2 & 2\left(q_x q_y - q_w q_z\right) & 2\left(q_z q_x + q_w q_y\right) & 0 \\ 0 & 2\left(q_x q_y + q_w q_z\right) & 1 - 2q_z^2 - 2q_x^2 & 2\left(q_y q_z - q_w q_x\right) & 0 \\ 0 & 2\left(q_z q_x - q_w q_y\right) & 2\left(q_y q_z + q_w q_x\right) & 1 - 2q_x^2 - 2q_y^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Relativistic Quaternions

- Motor (dual quaternion) from 4D algebra becomes

$$\mathbf{d}(\tau) = \boldsymbol{l}\sin\left(\frac{1}{2}\gamma\tau\dot{\phi}\right) + \mathbb{1}\cos\left(\frac{1}{2}\gamma\tau\dot{\phi}\right) - \left(\frac{1}{2}\gamma\tau\dot{\delta}\boldsymbol{l}^{\star} \wedge \mathbf{e}_0\right)\cos\left(\frac{1}{2}\gamma\tau\dot{\phi}\right) - \frac{1}{2}\gamma\tau\dot{\delta}\mathbf{e}_0\sin\left(\frac{1}{2}\gamma\tau\dot{\phi}\right)$$

- Again multiply by temporal translation

$$\mathbf{D}(\tau) = \mathbf{d}(\tau) \dot{\vee} \left(\mathbb{1} - \frac{1}{2}\gamma c\tau\,\mathbf{e}_{321}\right)$$

Relativistic Quaternions

$$\mathbf{D}(\tau) = \boldsymbol{l}\sin\left(\frac{1}{2}\gamma\tau\dot{\phi}\right) + \mathbb{1}\cos\left(\frac{1}{2}\gamma\tau\dot{\phi}\right) - \left(\frac{1}{2}\gamma\tau\dot{\delta}\boldsymbol{l}^{\star} \wedge \mathbf{e}_0\right)\cos\left(\frac{1}{2}\gamma\tau\dot{\phi}\right) - \frac{1}{2}\gamma\tau\dot{\delta}\mathbf{e}_0\sin\left(\frac{1}{2}\gamma\tau\dot{\phi}\right)$$
$$- \frac{1}{2}\gamma c\tau\left[\boldsymbol{l}\sin\left(\frac{1}{2}\gamma\tau\dot{\phi}\right) + \mathbb{1}\cos\left(\frac{1}{2}\gamma\tau\dot{\phi}\right)\right] \vee \mathbf{e}_{321}$$

$$\mathbf{D} = q_x\mathbf{e}_{410} + q_y\mathbf{e}_{420} + q_z\mathbf{e}_{430} + q_w\mathbb{1} + m_x\mathbf{e}_{230} + m_y\mathbf{e}_{310} + m_z\mathbf{e}_{120} + m_w\mathbf{e}_0$$
$$+ s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + s_w\mathbf{e}_{321}$$

Relativistic Quaternions

- Assuming weight one, norm of a relativistic dual quaternion is

$$\|\mathbf{D}\|_{\bullet} = \sqrt{\mathbf{D} \bullet \mathbf{D}} = \sqrt{s_x^2 + s_y^2 + s_z^2 + s_w^2 - m_x^2 - m_y^2 - m_z^2 - m_w^2}$$

- $m_x^2 + m_y^2 + m_z^2 + m_w^2$ is square of half the distance a particle starting at the origin is moved by **D** through space
- $s_x^2 + s_y^2 + s_z^2 + s_w^2 = \frac{1}{4}\gamma^2 c^2 \tau^2$ is square of half the distance through time

Relativistic Quaternions

- Norm is real precisely when particle moves at less than or equal to the speed of light

$$\left\|\mathbf{D}\right\|_{\bullet} = \sqrt{\mathbf{D} \bullet \mathbf{D}} = \sqrt{s_x^2 + s_y^2 + s_z^2 + s_w^2 - m_x^2 - m_y^2 - m_z^2 - m_w^2}$$

Relativistic Quaternions

- Setting $\dot{\phi} = 0$, **D** reduces to translation operator with norm

$$\left\| \mathbf{T}(\tau) \right\|_{\bullet} = \frac{1}{2}\gamma\tau\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$$

- Setting $dt = \gamma\tau$, $dx = \dot{x}\,dt$, $dy = \dot{y}\,dt$, and $dz = \dot{z}\,dt$,

$$\left\| \mathbf{T}(\tau) \right\|_{\bullet} = \frac{1}{2}\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$$

$$\mathbf{G} = \left[\begin{array}{c|ccccc|cccccccccc|cccccccccc|ccccc|c}
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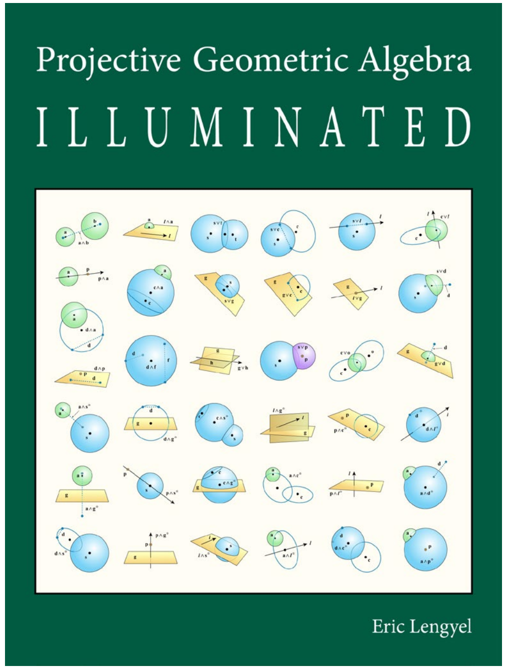
\end{array}\right]$$

References

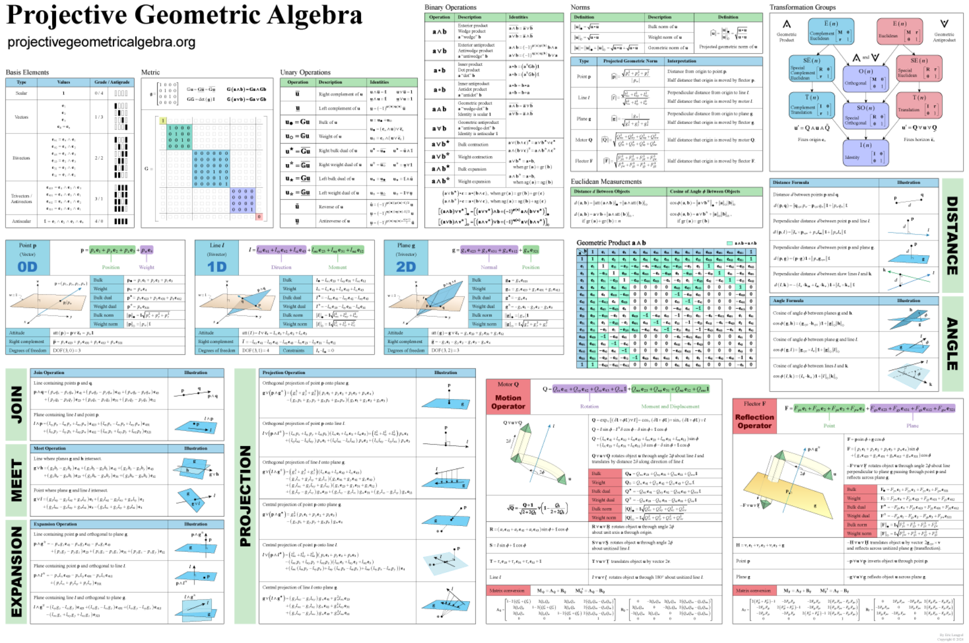
- Projective Geometric Algebra Illuminated
- projectivegeometricalgebra.org



Projective Geometric Algebra

ILLUMINATED

Eric Lengyel



Projective Geometric Algebra

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