Geometry and Motion in Projective Exterior and Geometric Algebras

Eric Lengyel, Ph.D.

Virginia Tech

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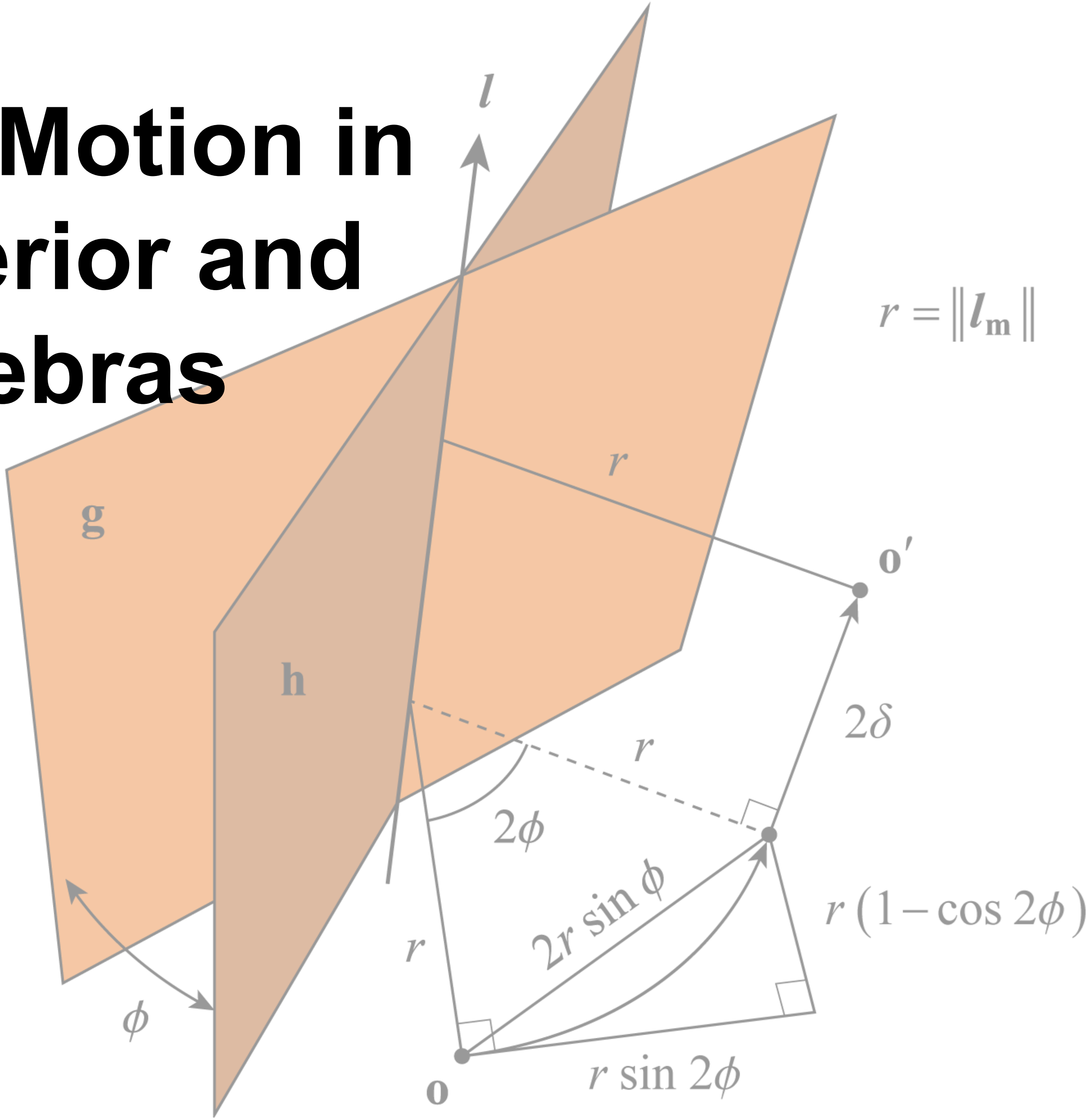
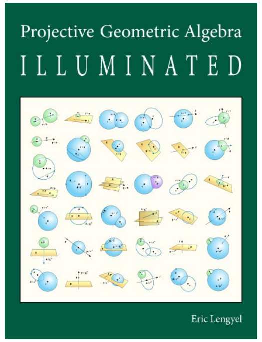
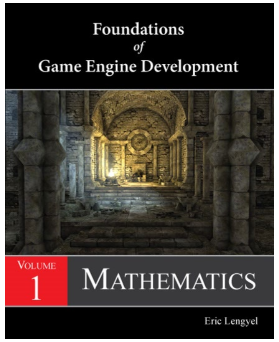
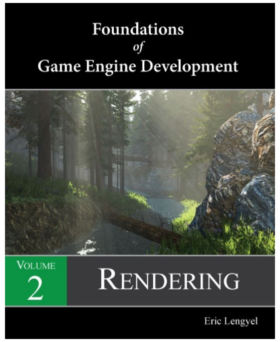


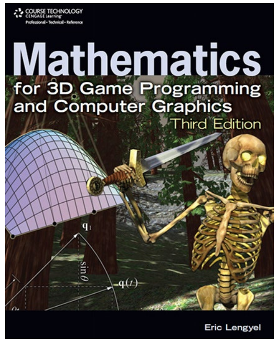
About the Speaker

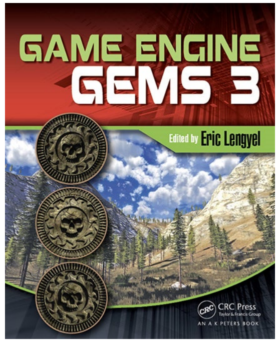
- Virginia Tech mathematics alumnus, B.S. 1994, M.S. 1996
- UC Davis computer science, Ph.D. 2010
- Working in industry since 1994 (former Sierra, Apple, Sony)
- Developing algebraic models for about 15 years
- Occasionally teaches computer graphics
- Writes books about math and real-time rendering

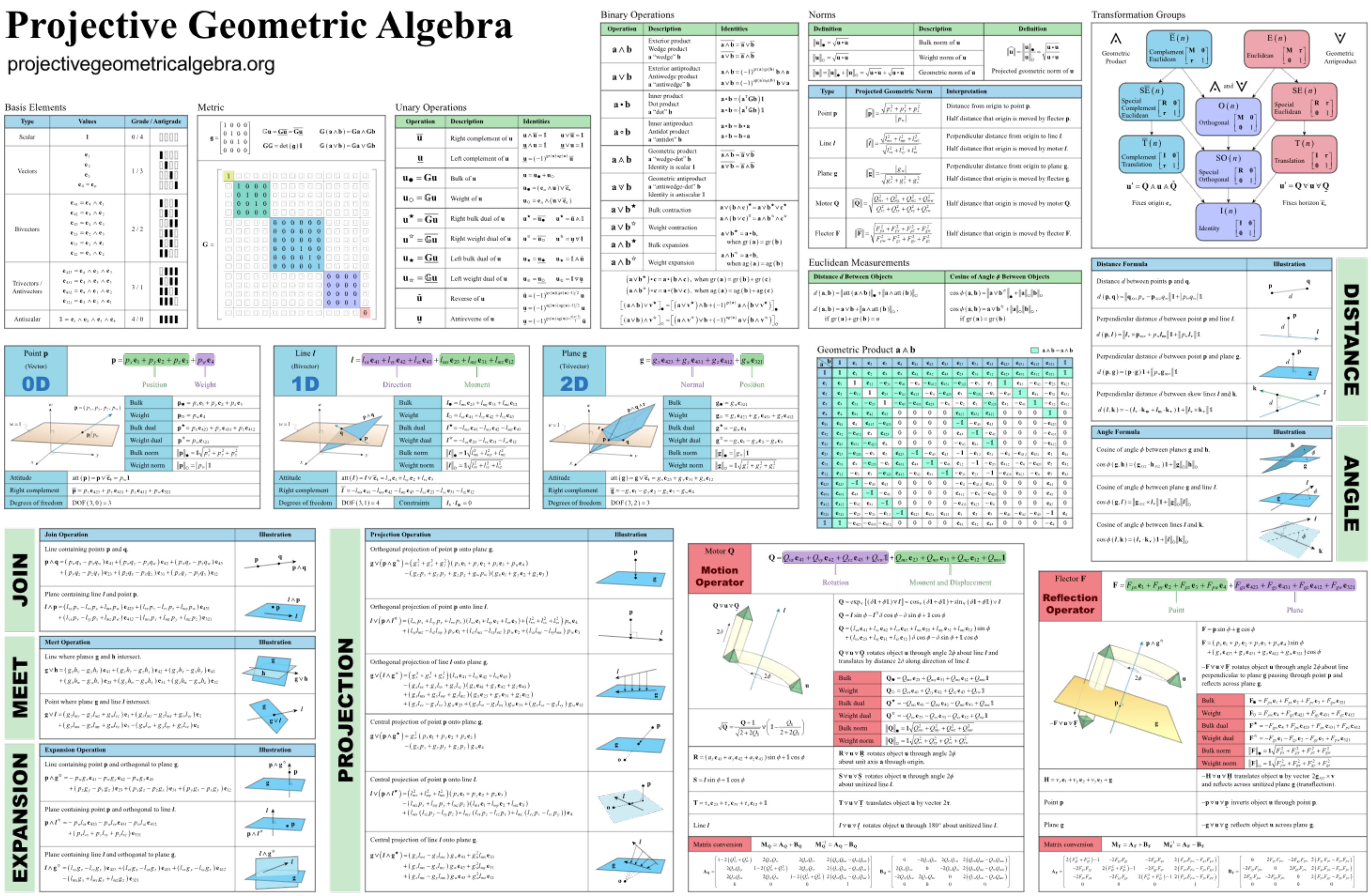








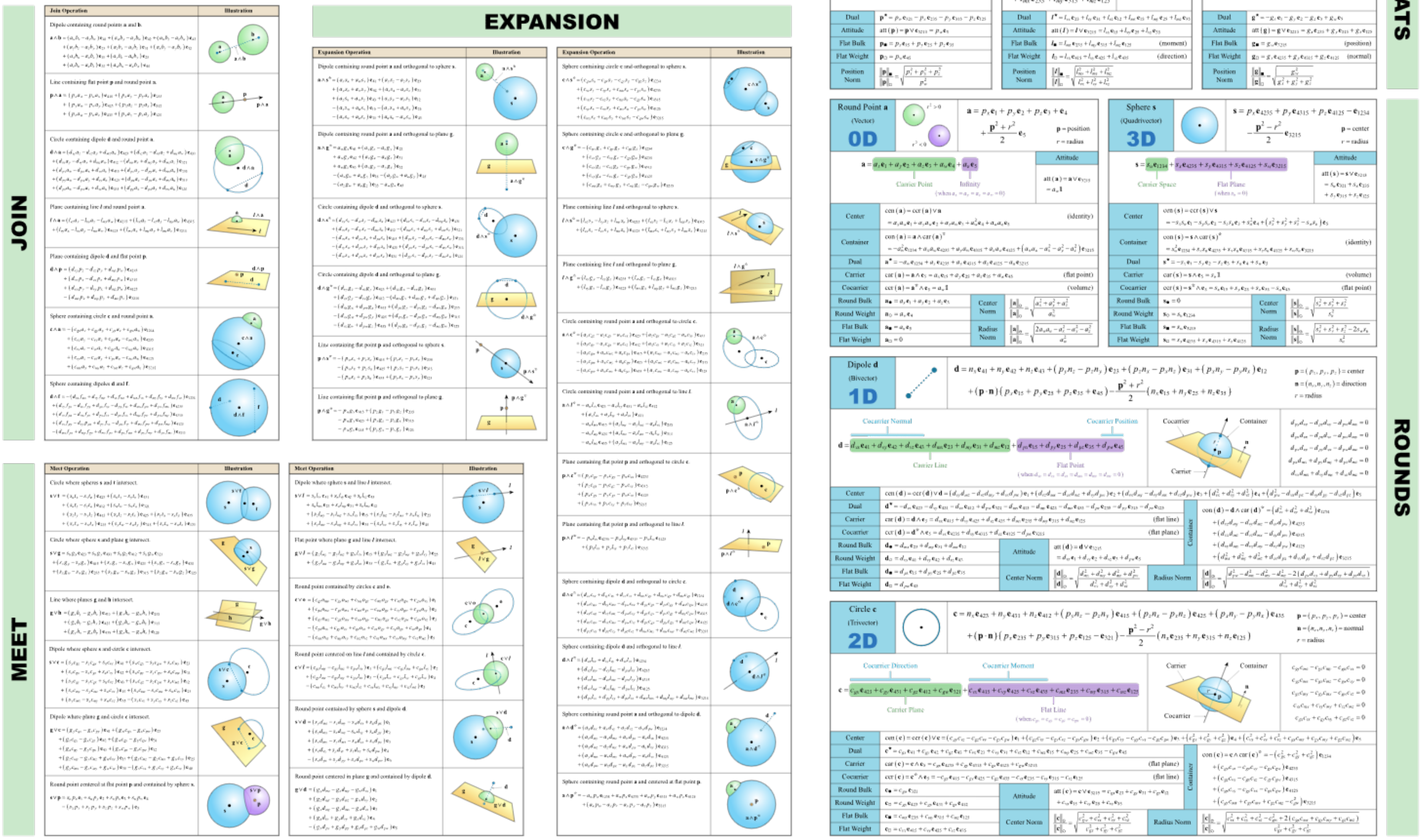




Projective Geometric Algebra
projectivegeometricalgebra.org
Basis Elements
Metric
Unary Operations
Binary Operations
Norms
Transformation Groups
Euclidean Measurements
Geometric Product
Point p
0D
Line l
1D
Plane g
2D
JOIN
MEET
EXPANSION
PROJECTION
Motor Q
Motion Operator
Flector F
Reflection Operator
DISTANCE
ANGLE

Conformal Geometric Algebra

conformalgeometricalgebra.org



A Vast Subject Area

- No hope of covering all the fundamentals in one hour

- This talk is an introduction that paints the big picture

Grassmann / Clifford Algebras

- You've probably been using pieces of these algebras already without realizing it

- Cross products
- Homogeneous coordinates (x, y, z, w)
- Planes (a, b, c, d)
- Plücker coordinates
- Quaternions

Cross Products

- Units of distance become units of area

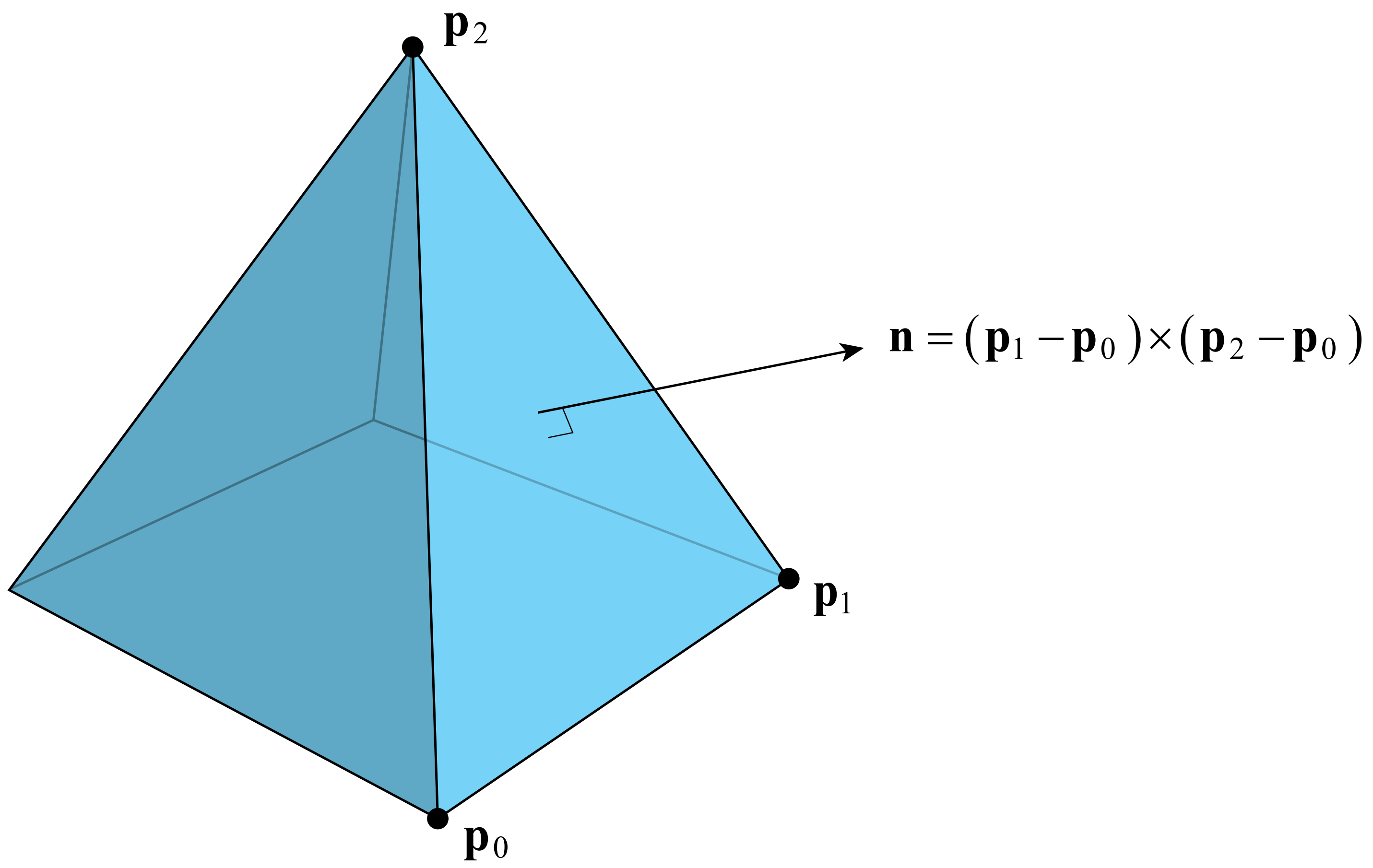
$$(a_x, a_y, a_z) \times (b_x, b_y, b_z)$$

$$\downarrow$$

$$(a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x)$$

Normal Vectors

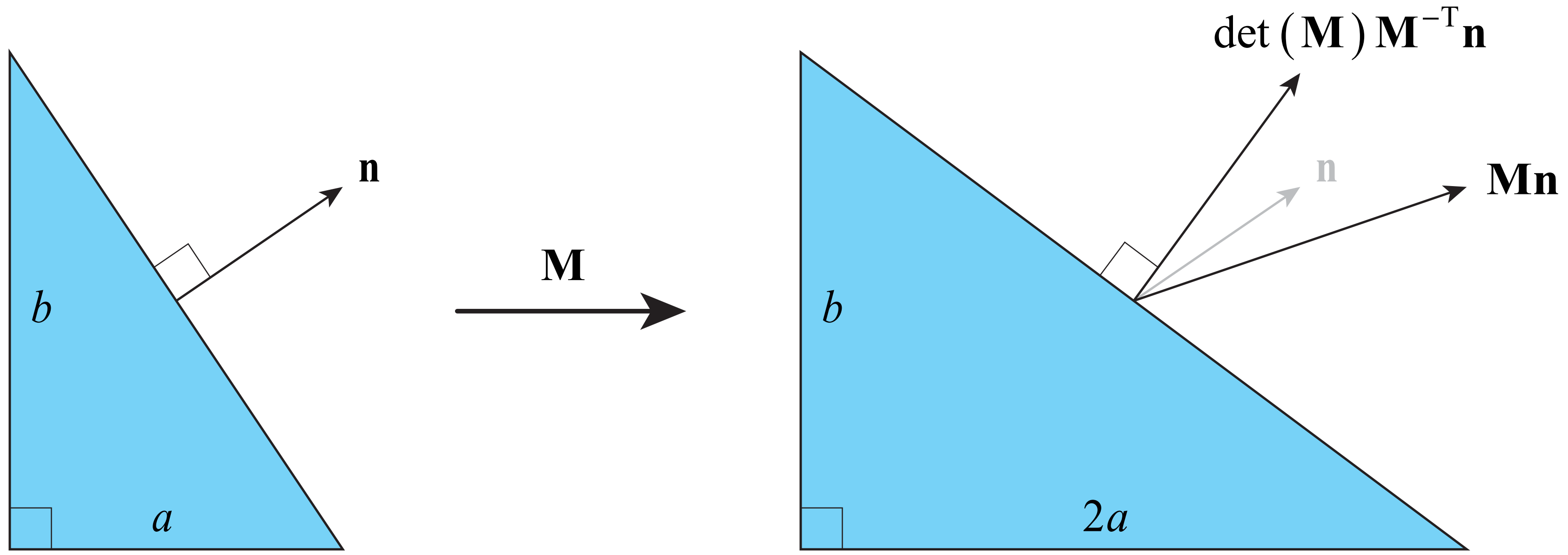
- Cross product calculates normal of triangular face



Normal Vector Transformation

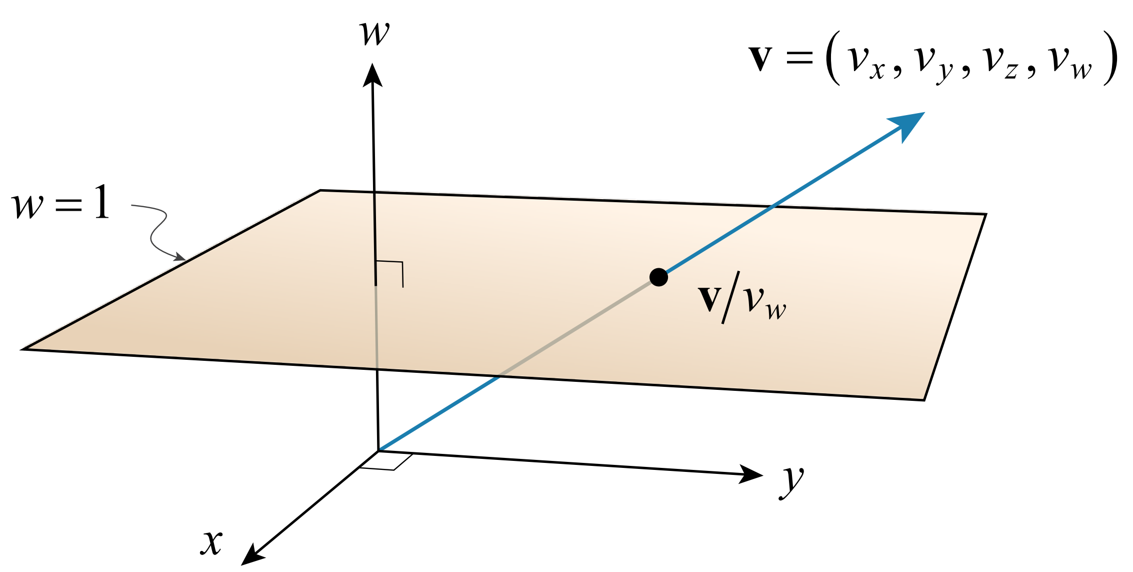
- Normals don’t transform like ordinary vectors
- That’s because they’re something else called *bivectors*

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Homogeneous Coordinates

- 3D points are projections of 4D vectors



Homogeneous Coordinates

- Allows translations to be added to linear transformations

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

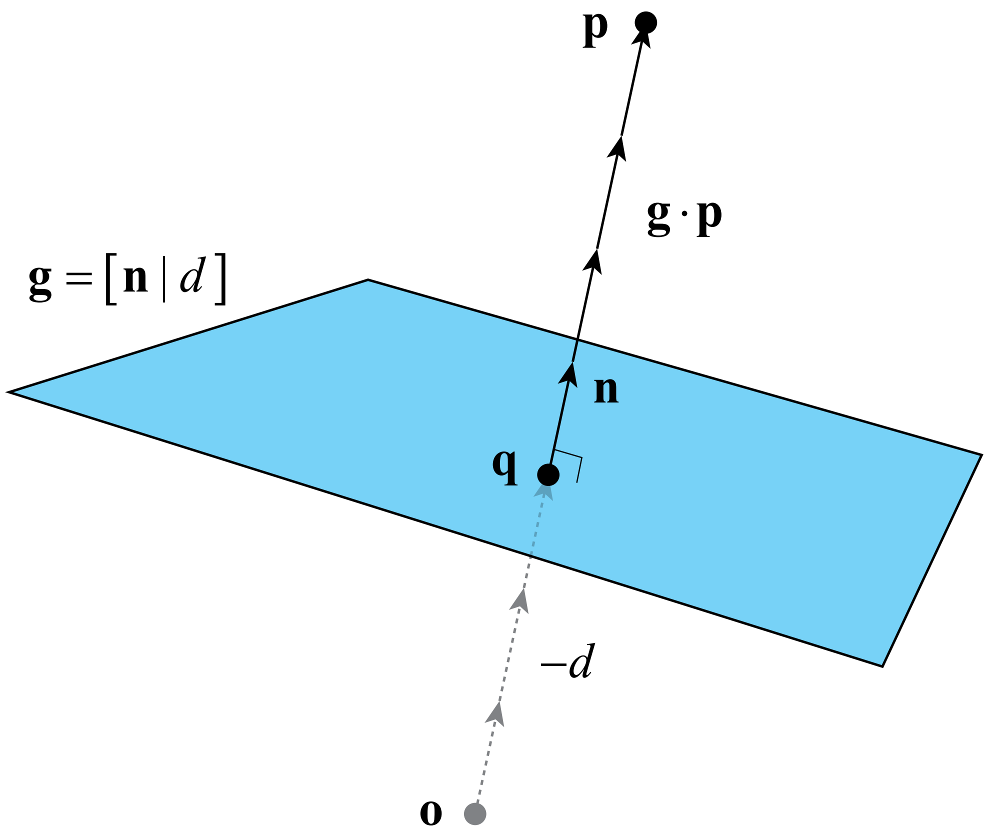
$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & t_x \\ m_{21} & m_{22} & m_{23} & t_y \\ m_{31} & m_{32} & m_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Planes

- 4D dot product with point **p** gives signed distance to plane **g**

$$\mathbf{p} = (x, y, z, w)$$

$$\mathbf{g} = (n_x, n_y, n_z, d)$$



Plücker Coordinates

- Implicit representation of a line in 3D space
- Has 6 coordinates, 3 for direction **v** and 3 for moment **m**

- Given homogeneous points **p** and **q** on the line,

$$\mathbf{v} = p_w \mathbf{q}_{xyz} - q_w \mathbf{p}_{xyz}$$

$$\mathbf{m} = \mathbf{p}_{xyz} \times \mathbf{q}_{xyz}$$

- Same results for any two points spaced same distance apart
- Information about specific points is eliminated

Points, Lines, Planes

- Lots of formulas for combining geometries
- Discovered without knowledge of bigger picture
- We can better explain where all of these formulas come from

	Formula	Description
A	$\{w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two homogeneous points $(\mathbf{p}_1 \mid w_1)$ and $(\mathbf{p}_2 \mid w_2)$.
B	$\{\mathbf{p}_2 - \mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two points $\mathbf{p}_1$ and $\mathbf{p}_2$.
C	$\{\mathbf{v} \mid \mathbf{p} \times \mathbf{v}\}$	Line through point $\mathbf{p}$ with direction $\mathbf{v}$.
D	$\{\mathbf{p} \mid \mathbf{0}\}$	Line through point $\mathbf{p}$ and the origin.
E	$[\mathbf{v} \times \mathbf{p} + w\mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and homogeneous point $(\mathbf{p} \mid w)$.
F	$[\mathbf{v} \times \mathbf{p} + \mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and point $\mathbf{p}$.
G	$[\mathbf{v} \times \mathbf{u} \mid -\mathbf{u} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$, parallel to direction $\mathbf{u}$.
H	$[\mathbf{m} \mid 0]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and the origin.
I	$\{\mathbf{n}_1 \times \mathbf{n}_2 \mid d_1\mathbf{n}_2 - d_2\mathbf{n}_1\}$	Line where two planes $[\mathbf{n}_1 \mid d_1]$ and $[\mathbf{n}_2 \mid d_2]$ intersect.
J	$(\mathbf{m} \times \mathbf{n} + d\,\mathbf{v} \mid -\mathbf{n} \cdot \mathbf{v})$	Homogeneous point where line $\{\mathbf{v} \mid \mathbf{m}\}$ intersects plane $[\mathbf{n} \mid d]$.
K	$\{w\mathbf{n} \mid \mathbf{p} \times \mathbf{n}\}$	Line through homogeneous point $(\mathbf{p} \mid w)$, perpendicular to plane $[\mathbf{n} \mid d]$.
L	$[\mathbf{v} \times \mathbf{n} \mid -\mathbf{n} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$, perpendicular to plane $[\mathbf{n} \mid d]$.
M	$[w\mathbf{v} \mid -\mathbf{p} \cdot \mathbf{v}]$	Plane containing homogeneous point $(\mathbf{p} \mid w)$, perpendicular to line $\{\mathbf{v} \mid \mathbf{m}\}$.
N	$(\mathbf{v} \times \mathbf{m} \mid \mathbf{v}^2)$	Homogeneous point closest to the origin on line $\{\mathbf{v} \mid \mathbf{m}\}$.
O	$(-d\mathbf{n} \mid \mathbf{n}^2)$	Homogeneous point closest to the origin on plane $[\mathbf{n} \mid d]$.
P	$[\mathbf{m} \times \mathbf{v} \mid \mathbf{m}^2]$	Plane farthest from the origin containing line $\{\mathbf{v} \mid \mathbf{m}\}$.
Q	$[-w\mathbf{p} \mid \mathbf{p}^2]$	Plane farthest from the origin containing point $(\mathbf{p} \mid w)$.
R	$\dfrac{\lVert w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \rVert}{\lvert w_1 w_2 \rvert}$	Distance between two homogeneous points $(\mathbf{p}_1 \mid w_1)$ and $(\mathbf{p}_2 \mid w_2)$.
S	$\dfrac{\lvert \mathbf{v}_1 \cdot \mathbf{m}_2 + \mathbf{v}_2 \cdot \mathbf{m}_1 \rvert}{\lVert \mathbf{v}_1 \times \mathbf{v}_2 \rVert}$	Distance between two lines $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ and $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$.
T	$\dfrac{\lVert \mathbf{v} \times \mathbf{p} + \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to point $\mathbf{p}$.
U	$\dfrac{\lVert \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to the origin.
V	$\dfrac{\lvert \mathbf{n} \cdot \mathbf{p} + d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to point $\mathbf{p}$.
W	$\dfrac{\lvert d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to the origin.

Quaternions

- A quaternion **q** represents a rotation in 3D space

$$\mathbf{q} = xi + yj + zk + w \qquad i^2 = j^2 = k^2 = -1 \qquad \begin{aligned} ij &= -ji = k \\ jk &= -kj = i \\ ki &= -ik = j \end{aligned}$$

- Rotation through angle ϕ about axis **a** is

$$\mathbf{q} = \left(\sin\frac{\phi}{2} \right) \mathbf{a} + \cos\frac{\phi}{2}$$

Quaternions

- A quaternion rotates a vector **v** with the sandwich product

$$\mathbf{v}' = \mathbf{q}\mathbf{v}\mathbf{q}^{*} \qquad \mathbf{v} = v_x i + v_y j + v_z k$$

- **q*** is the conjugate of the quaternion:

$$\mathbf{q} = -xi - yj - zk + w$$

All Part of Same Algebraic Structure

- Non-vector result of cross product
- 4D homogeneous coordinates for points
- 6D Plücker coordinates for lines
- 4D plane representations
- Quaternions

4D Associative Projective Algebras

- 4D rigid exterior algebra
  - Homogeneous representation of 3D geometry
  - Points, lines, planes
  - Join, meet, projection, norm, distance, angle

- 4D rigid geometric algebra
  - Euclidean isometries in 3D space
  - Rotations, translations, screw transformations
  - Parameterization, interpolation

Exterior / Grassmann Algebra

- Wedge product $\wedge$
  - Combines dimensions of operands
  - Vectors square to zero:

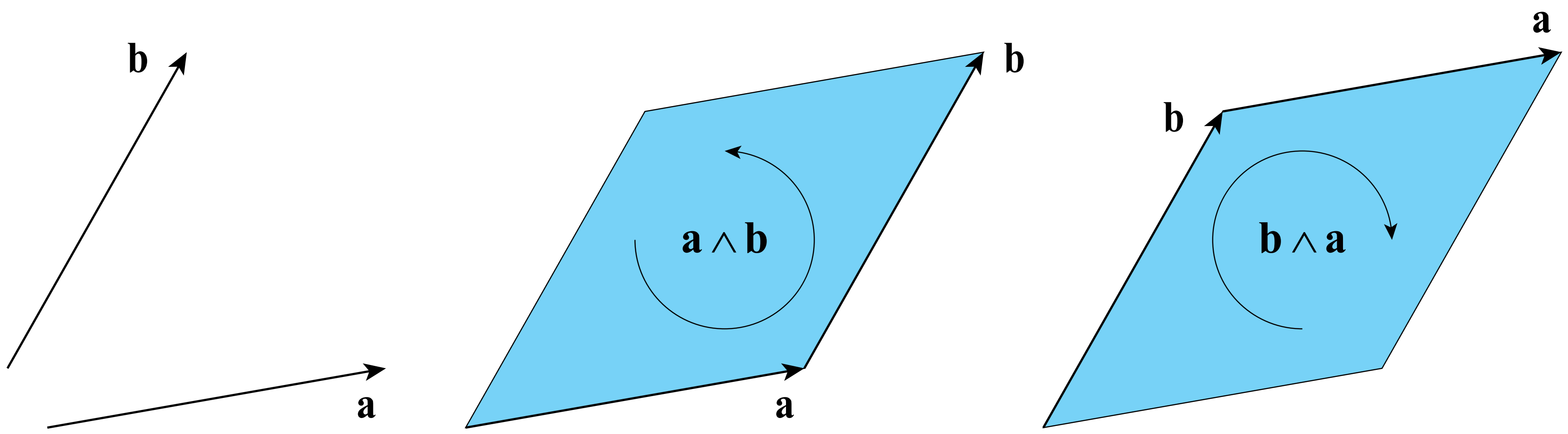
$$\mathbf{v} \wedge \mathbf{v} = 0$$

  - Antisymmetric on vectors:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

Bivectors

- Wedge product of two vectors **a** and **b**
- Produces a new type of object



Bivectors

- Wedge product of two vectors **a** and **b**:

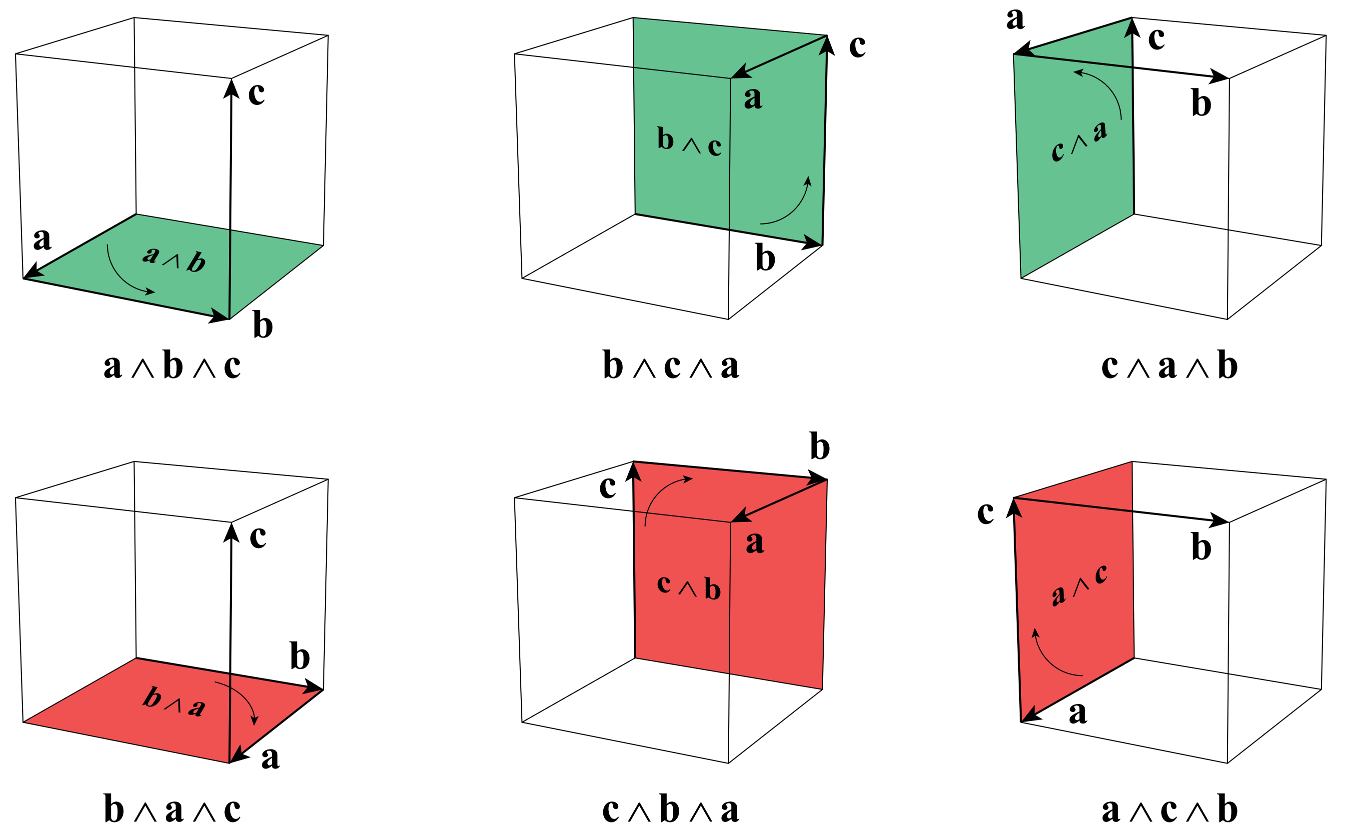
$$\begin{aligned}\mathbf{a}\wedge\mathbf{b} &= \left(a_y b_z - a_z b_y\right)\left(\mathbf{e}_2\wedge\mathbf{e}_3\right)\\ &+\left(a_z b_x - a_x b_z\right)\left(\mathbf{e}_3\wedge\mathbf{e}_1\right)\\ &+\left(a_x b_y - a_y b_x\right)\left(\mathbf{e}_1\wedge\mathbf{e}_2\right)\end{aligned}$$

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b} &= \left(a_y b_z - a_z b_y\right)\mathbf{e}_{23}\\ &+\left(a_z b_x - a_x b_z\right)\mathbf{e}_{31}\\ &+\left(a_x b_y - a_y b_x\right)\mathbf{e}_{12}\end{aligned}$$

- Cross product appears!

Trivectors

- Wedge product of three vectors **a**, **b**, and **c**



Trivectors

- Wedge product of three vectors **a**, **b**, and **c**

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \left(a_x b_y c_z + a_y b_z c_x + a_z b_x c_y - a_x b_z c_y - a_y b_x c_z - a_z b_y c_x \right) \mathbf{e}_{123}$$

- Determinant of 3 × 3 matrix with columns **a**, **b**, and **c**

Trivectors

- Wedge product of vector **a** and bivector **b**

$$\mathbf{a} = a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3$$

$$\mathbf{b} = b_x \mathbf{e}_{23} + b_y \mathbf{e}_{31} + b_z \mathbf{e}_{12}$$

$$\mathbf{a} \wedge \mathbf{b} = \left(a_x b_x + a_y b_y + a_z b_z \right) \mathbf{e}_{123}$$

- Dot product appears!

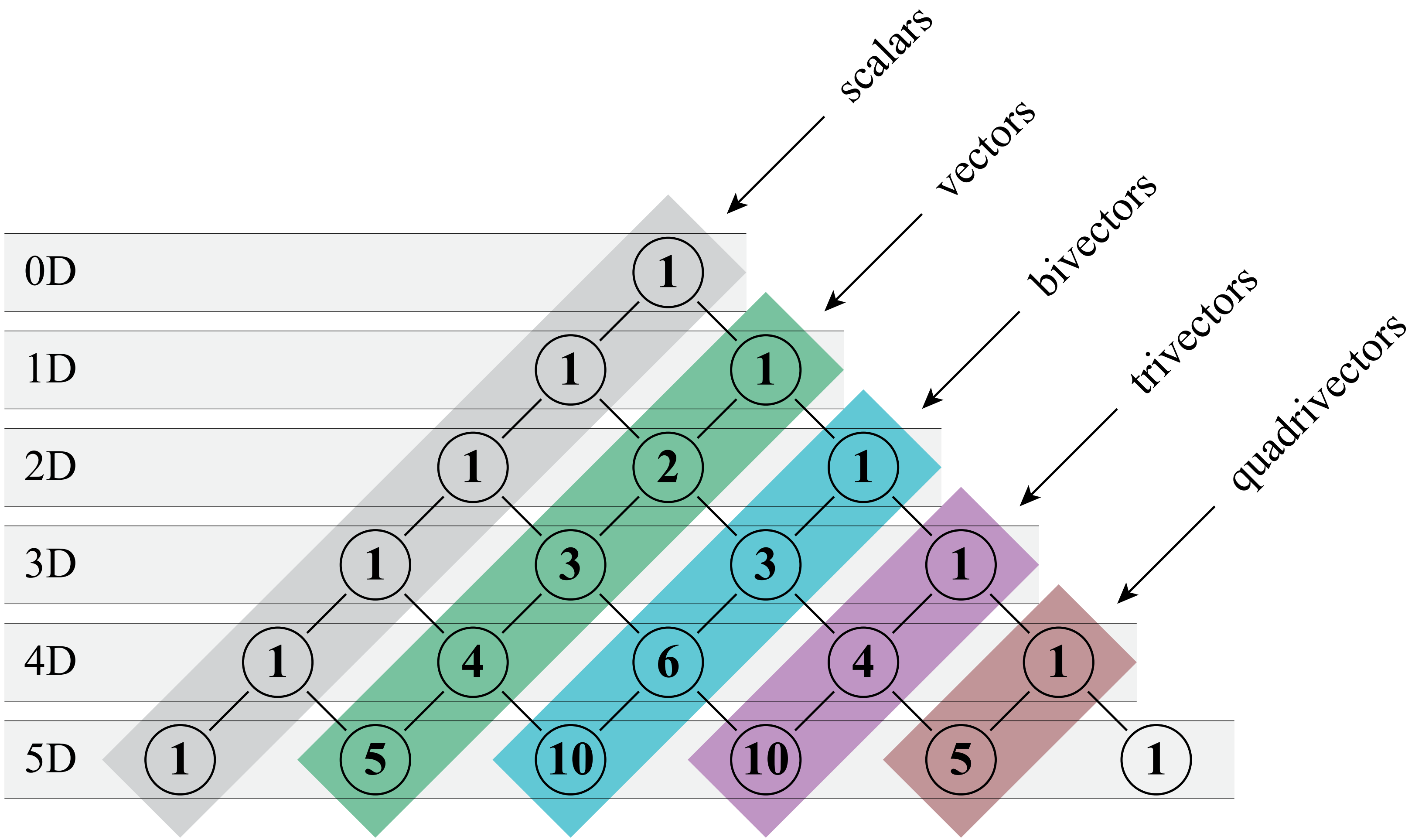
3D Vector Space

Scalars	s	Magnitudes
Vectors	$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$	Directed lengths

3D Exterior Algebra

Scalars	$s\mathbf{1}$	Magnitudes
Vectors	$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$	Directed lengths
Bivectors	$x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12}$	Directed areas
Trivectors	$t\mathbf{e}_{123}$	Directed volumes

Pascal's Triangle



Rigid Exterior / Geometric Algebra

- Projective algebra with one extra dimension
- Contains points, lines, planes in 3D
- Can perform rotations, translations, screw transformations

4D Exterior Algebra

- Extends 4D vector space

- One scalar **1**
- Four vector basis elements
- Six bivector basis elements
- Four trivector basis elements
- One antiscalar $\mathbb{1}$

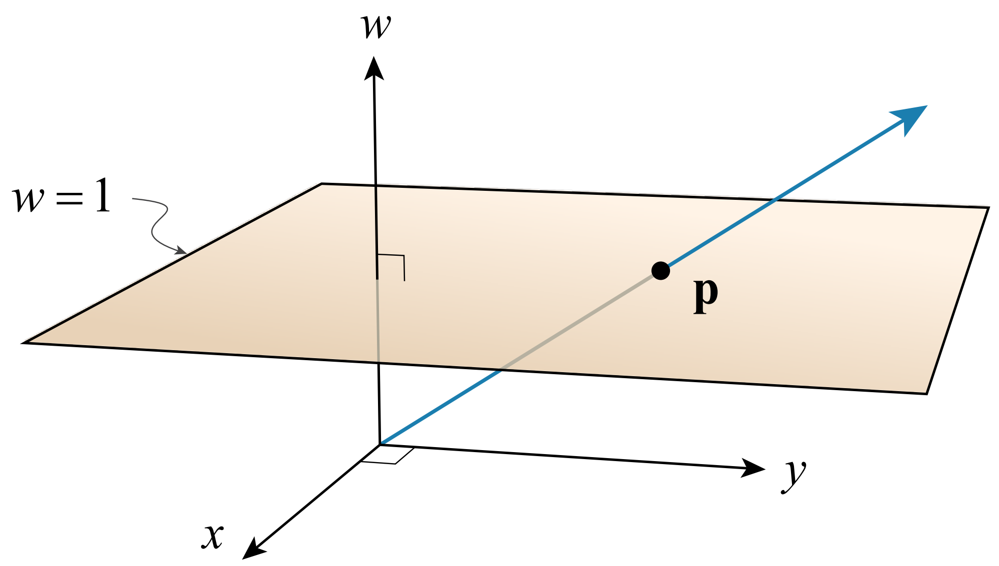
Type	Values	Grade / Antigrade	
Scalar	$\mathbf{1}$	0 / 4	▯▯▯▯
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4 = \mathbf{e}_n$	1 / 3	▮▯▯▯ ▯▮▯▯ ▯▯▮▯ ▯▯▯▮
Bivectors	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$	2 / 2	▮▯▯▮ ▯▮▯▮ ▯▯▮▮ ▯▮▮▯ ▮▯▮▯ ▮▮▯▯
Trivectors / Antivectors	$\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$	3 / 1	▯▮▮▮ ▮▯▮▮ ▮▮▯▮ ▮▮▮▯
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	▮▮▮▮

Point

$$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$$

Position ($p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$)

Weight ($p_w\mathbf{e}_4$)



Special Points

- The origin is simply the point $\mathbf{e}_4$
- Point with zero weight lies at infinity in (x, y, z) direction
- Points at infinity in opposite directions are equivalent

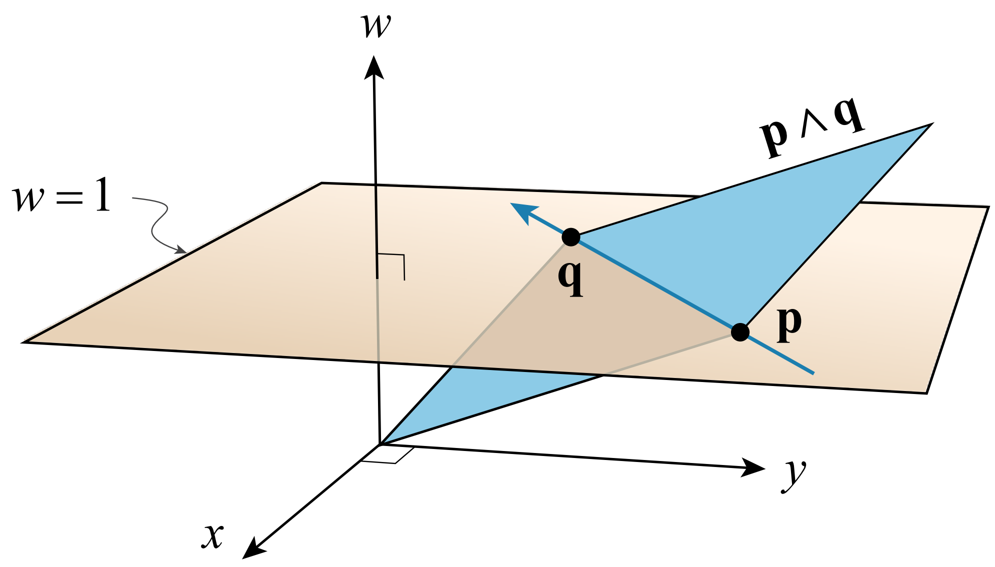
Line

$$\mathbf{p} \wedge \mathbf{q} = (q_x p_w - p_x q_w)\mathbf{e}_{41} + (q_y p_w - p_y q_w)\mathbf{e}_{42} + (q_z p_w - p_z q_w)\mathbf{e}_{43}$$
$$+ (p_y q_z - p_z q_y)\mathbf{e}_{23} + (p_z q_x - p_x q_z)\mathbf{e}_{31} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$$

$$\boldsymbol{l} = l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43} + l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}$$

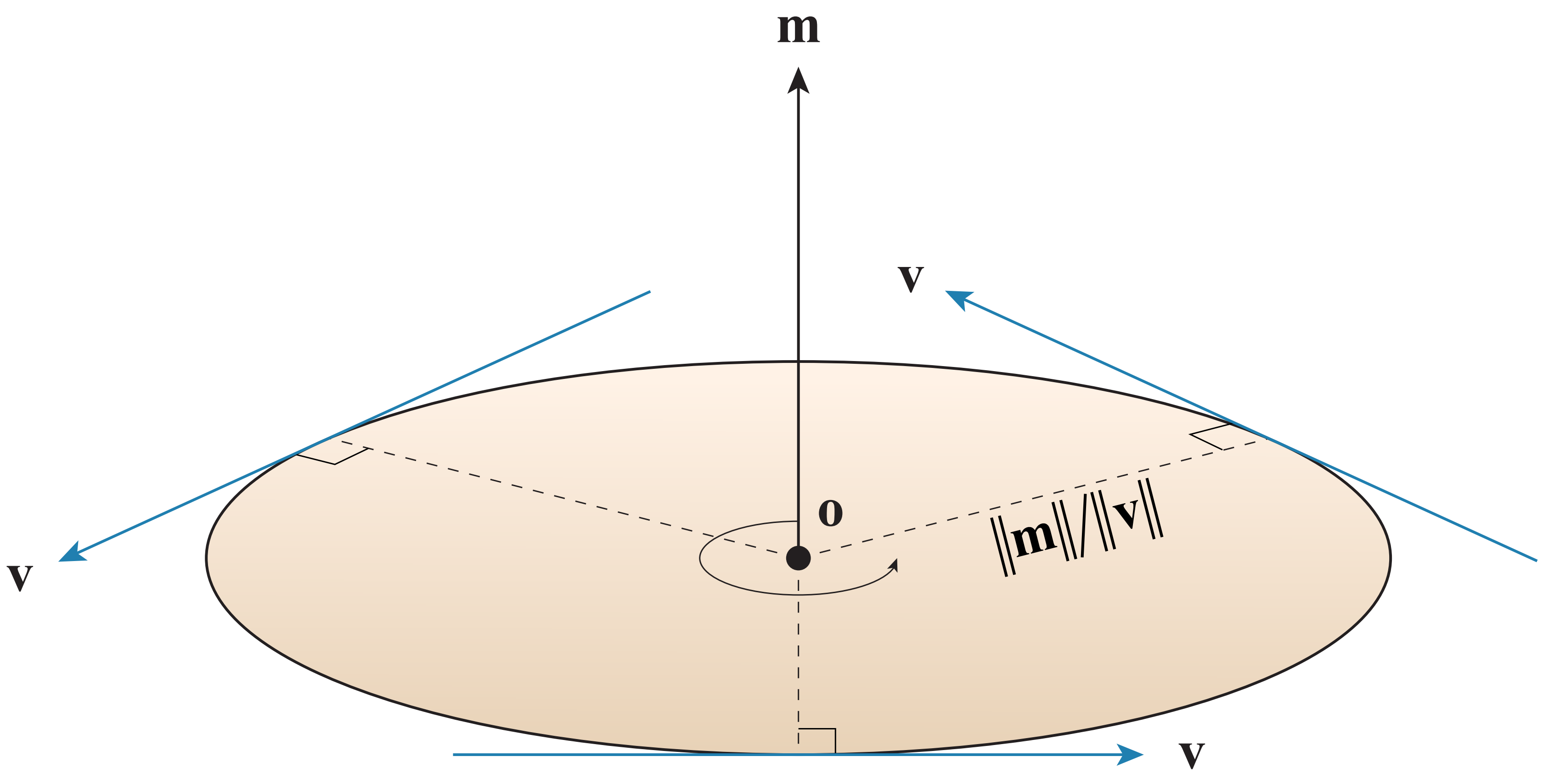
Direction (the $\mathbf{e}_{41}$, $\mathbf{e}_{42}$, $\mathbf{e}_{43}$ terms) — Moment (the $\mathbf{e}_{23}$, $\mathbf{e}_{31}$, $\mathbf{e}_{12}$ terms)

$$\boldsymbol{l}_\mathbf{v} \cdot \boldsymbol{l}_\mathbf{m} = 0$$



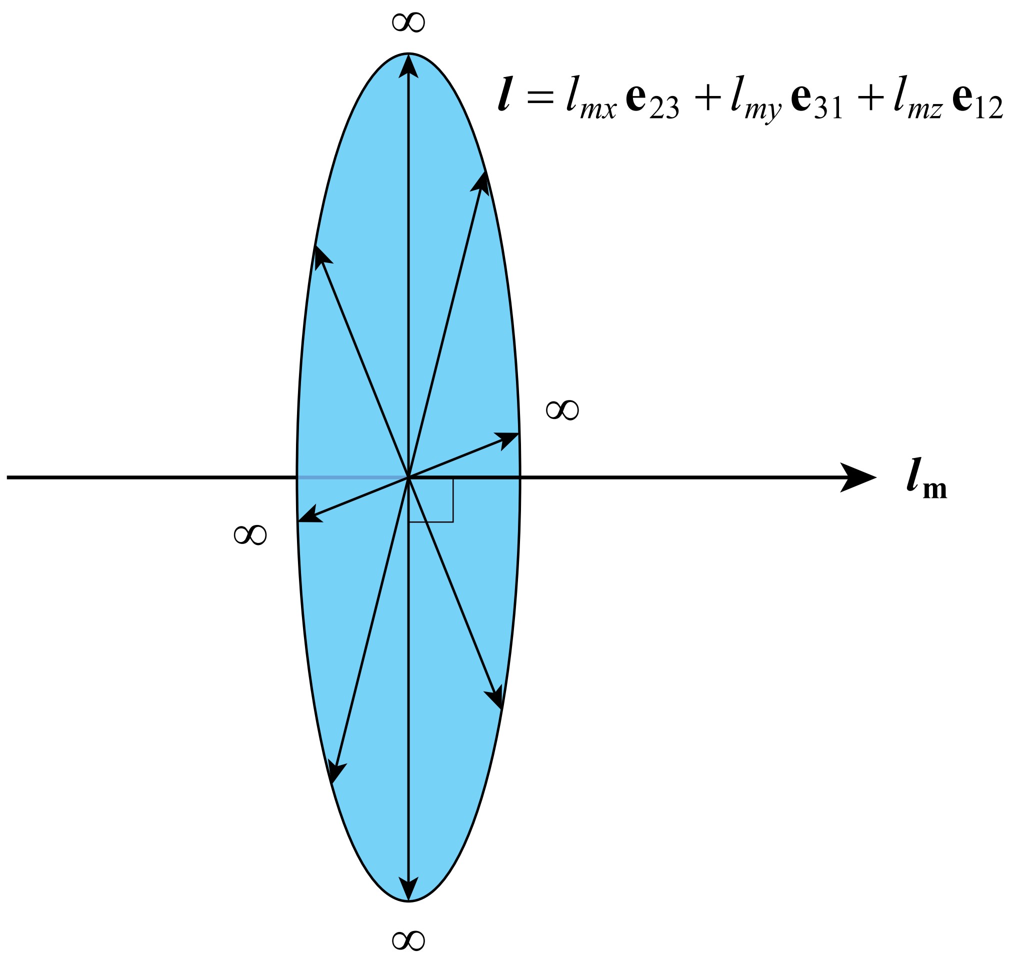
Line Moment

- Contains position information



Lines at Infinity

- Line with zero direction lies at infinity

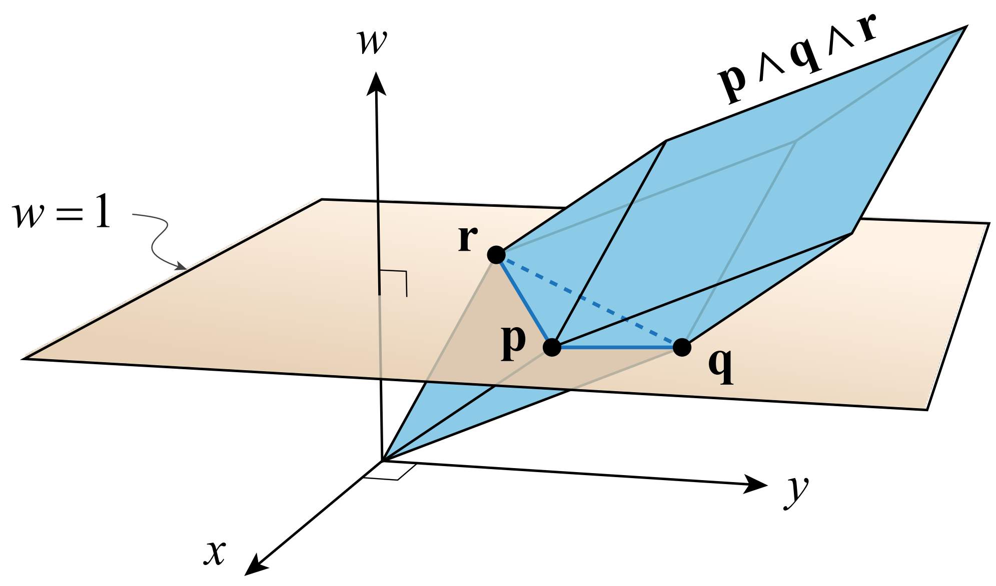


Plane

$$\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx})\,\mathbf{e}_{423} + (l_{vz} p_x - l_{vx} p_z + l_{my})\,\mathbf{e}_{431}$$
$$+ (l_{vx} p_y - l_{vy} p_x + l_{mz})\,\mathbf{e}_{423} - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\,\mathbf{e}_{321}$$

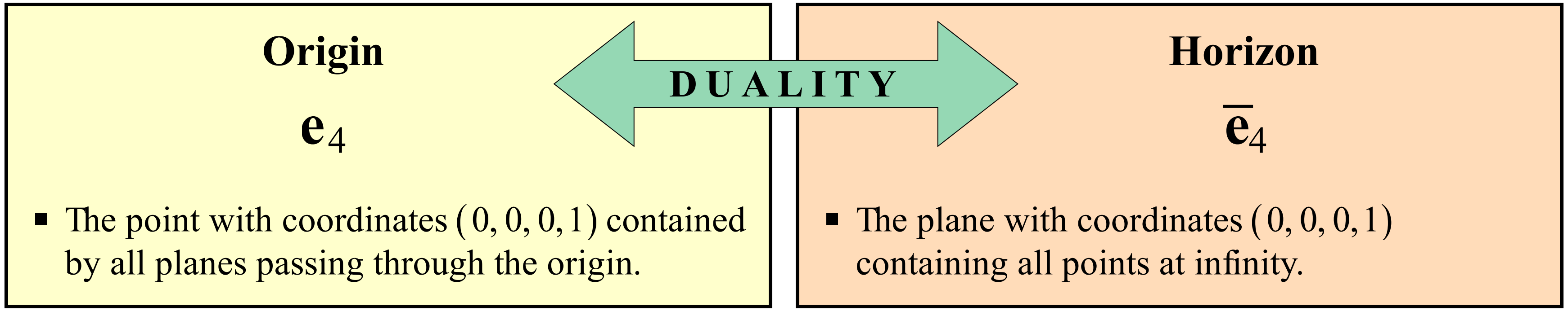
$$\mathbf{g} = g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} + g_w \mathbf{e}_{321}$$

Normal — Position



Horizon

- Plane with zero normal lies at infinity: $g_w \mathbf{e}_{321}$
- Contains all points at infinity, all lines at infinity
- Given special name *horizon*



4D Exterior Algebra

Scalars	$s\mathbf{1}$	Magnitudes
Vectors	$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_4$	Points
Bivectors	$v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	Lines
Trivectors	$g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321}$	Planes
Quadrivectors	$t\mathbb{1}$	Magnitudes

Complements

- Complement inverts full / empty dimensions
- Right complement denoted by overbar
- Left complement denoted by underbar
- For basis element **u**,

$$\mathbf{u} \wedge \overline{\mathbf{u}} = \mathbb{1} \qquad \underline{\mathbf{u}} \wedge \mathbf{u} = \mathbb{1}$$

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\overline{\mathbf{u}}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$\mathbf{1}$
$\underline{\mathbf{u}}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{1}$

Antiwedge Product

- Antiwedge product denoted by ∨

Wedge Product

$\mathbf{a} \wedge \mathbf{b}$

- Combines the dimensions that are present in **a** and **b**.
- Result has grade $\mathrm{gr}(\mathbf{a}) + \mathrm{gr}(\mathbf{b})$.
- Result has antigrade $\mathrm{ag}(\mathbf{a}) + \mathrm{ag}(\mathbf{b}) - n$.

DUALITY

Antiwedge Product

$\mathbf{a} \vee \mathbf{b}$

- Combines the dimensions that are absent in **a** and **b**.
- Result has grade $\mathrm{gr}(\mathbf{a}) + \mathrm{gr}(\mathbf{b}) - n$.
- Result has antigrade $\mathrm{ag}(\mathbf{a}) + \mathrm{ag}(\mathbf{b})$.

De Morgan Laws

- Every operation with ‘anti’ in its name satisfies a De Morgan law:

$$\overline{\mathbf{a} \vee \mathbf{b}} = \overline{\mathbf{a}} \wedge \overline{\mathbf{b}} \qquad \underline{\mathbf{a} \vee \mathbf{b}} = \underline{\mathbf{a}} \wedge \underline{\mathbf{b}}$$

- To calculate anti-operation,
  - Take a complement of each input
  - Perform the regular operation
  - Take opposite complement of the result

4D Exterior Product

Wedge Product $\mathbf{a} \wedge \mathbf{b}$

a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_1$	0	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	0	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0	0
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	0	$-\mathbf{e}_{423}$	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	0	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	0	0	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0
$\mathbf{e}_4$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	0	$\mathbf{e}_{423}$	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	0	0	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	0	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}$	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

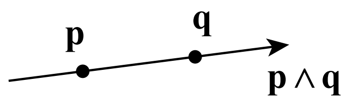
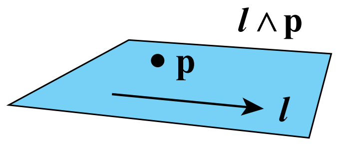
4D Exterior Antiproduct

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	$\mathbf{e}_3$
$\mathbf{e}_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	$\mathbf{e}_4$
$\mathbf{e}_{41}$	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	0	0	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	0	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	0	0	0	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	0	$-\mathbf{1}$	0	0	0	0	0	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	0	0	$-\mathbf{1}$	0	0	0	0	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	0	0	0	$-\mathbf{1}$	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	0	0	0	$-\mathbf{e}_3$	$\mathbf{e}_2$	0	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	0	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_3$	0	$-\mathbf{e}_1$	$\mathbf{e}_{43}$	0	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	0	0	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$-\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

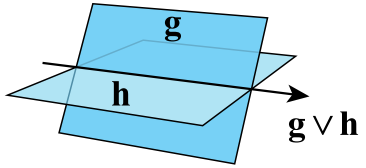
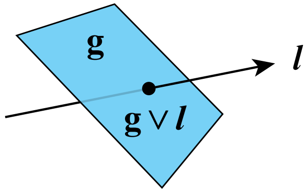
Join

- Wedge product performs join operation
- Produces higher-dimensional object containing both operands

Join Operation	**Illustration**
Line containing points **p** and **q**. $\mathbf{p} \wedge \mathbf{q} = (p_w q_x - p_x q_w)\,\mathbf{e}_{41} + (p_w q_y - p_y q_w)\,\mathbf{e}_{42} + (p_w q_z - p_z q_w)\,\mathbf{e}_{43} + (p_y q_z - p_z q_y)\,\mathbf{e}_{23} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}$	
Plane containing line ***l*** and point **p**. $\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx} p_w)\,\mathbf{e}_{423} + (l_{vz} p_x - l_{vx} p_z + l_{my} p_w)\,\mathbf{e}_{431} + (l_{vx} p_y - l_{vy} p_x + l_{mz} p_w)\,\mathbf{e}_{412} - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\,\mathbf{e}_{321}$	

Meet

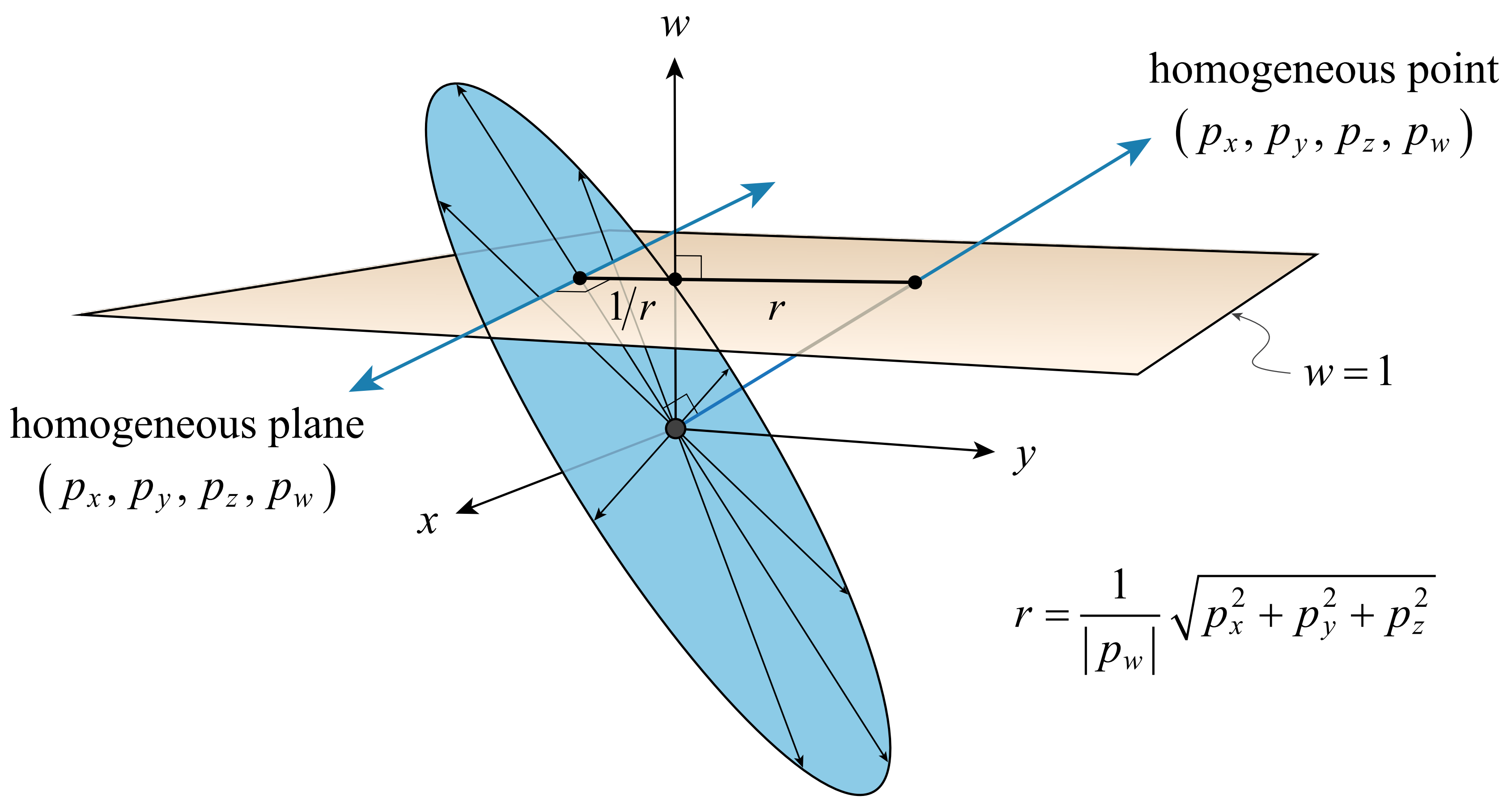
- Antiwedge product performs meet operation
- Produces lower-dimensional object at intersection of operands

Meet Operation	Illustration
Line where planes **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\,\mathbf{e}_{41} + (g_x h_z - g_z h_x)\,\mathbf{e}_{42} + (g_y h_x - g_x h_y)\,\mathbf{e}_{43} + (g_x h_w - g_w h_x)\,\mathbf{e}_{23} + (g_y h_w - g_w h_y)\,\mathbf{e}_{31} + (g_z h_w - g_w h_z)\,\mathbf{e}_{12}$	
Point where plane **g** and line ***l*** intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\,\mathbf{e}_1 + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\,\mathbf{e}_2 + (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\,\mathbf{e}_3 - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\,\mathbf{e}_4$	

Duality

- Every object can be interpreted as two different things
- Every operation performs two different actions
- One interpretation corresponds to regular space
- The other interpretation corresponds to *antispace*

Duality



$$r = \frac{1}{|p_w|}\sqrt{p_x^2 + p_y^2 + p_z^2}$$

Exomorphisms

- Given an n x n linear transformation **m** that operates on vectors
- The exomorphism **M** is the 2^n x 2^n matrix that operates on the whole algebra
- Exomorphism preserves structure under the wedge product:

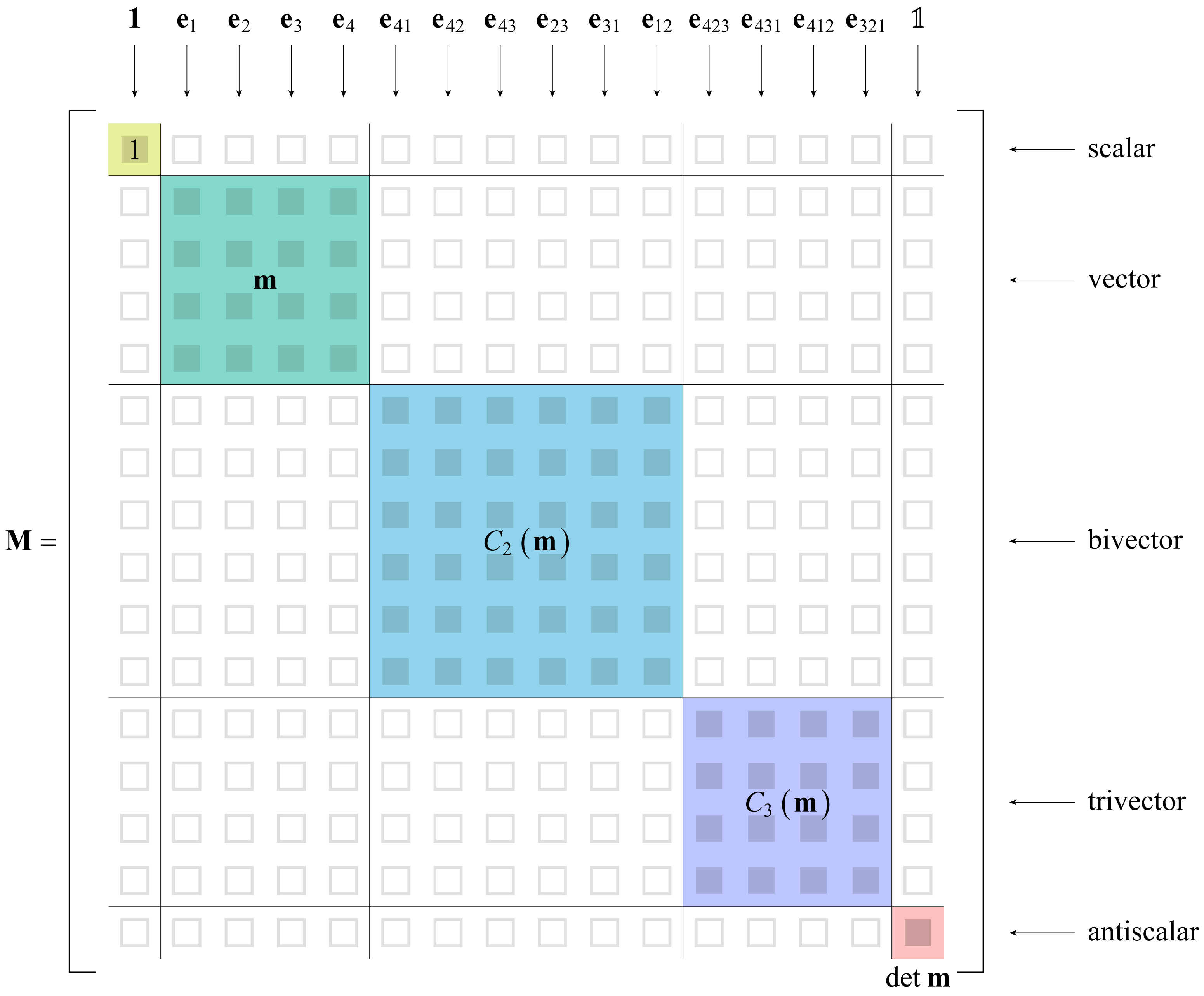
$$\mathbf{M}(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{M}\mathbf{a}) \wedge (\mathbf{M}\mathbf{b})$$

Exomorphisms

- Matrix **M** is block diagonal

- Each block has columns given by wedge products of columns of the original matrix **m**

- These are called *compound matrices* of **m**



1
e1
e2
e3
e4
e41
e42
e43
e23
e31
e12
e423
e431
e412
e321
𝟙
M =
1
m
C2 (m)
C3 (m)
det m
scalar
vector
bivector
trivector
antiscalar

Translation Exomorphism

$$\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -t_z & t_y & 1 & 0 & 0 \\ t_z & 0 & -t_x & 0 & 1 & 0 \\ -t_y & t_x & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_x & -t_y & -t_z & 1 \end{bmatrix}$$

Nonuniform Scale Exomorphism

$$\mathbf{m} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2(\mathbf{m}) = \begin{bmatrix} s_x & 0 & 0 & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 & 0 & 0 \\ 0 & 0 & s_z & 0 & 0 & 0 \\ 0 & 0 & 0 & s_y s_z & 0 & 0 \\ 0 & 0 & 0 & 0 & s_z s_x & 0 \\ 0 & 0 & 0 & 0 & 0 & s_x s_y \end{bmatrix}$$

$$C_3(\mathbf{m}) = \begin{bmatrix} s_y s_z & 0 & 0 & 0 \\ 0 & s_z s_x & 0 & 0 \\ 0 & 0 & s_x s_y & 0 \\ 0 & 0 & 0 & s_x s_y s_z \end{bmatrix}$$

The Metric Tensor

- n x n matrix that defines dot products of vectors

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$

$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_4 \cdot \mathbf{e}_4 = 0$$

$$\mathfrak{g}_{ij} \equiv \mathbf{v}_i \cdot \mathbf{v}_j$$

Metric Exomorphism

- The metric tensor is a linear transformation
- It can be extended to a $2^n \times 2^n$ matrix **G** that applies to entire exterior algebra
- There is also an *antimetric* that satisfies $\mathbb{G}\mathbf{u} = \underline{\mathbf{G}\bar{\mathbf{u}}} = \overline{\mathbf{G}\underline{\mathbf{u}}}$

Metric

$\mathbf{G}$

- Is an exomorphism such that $\mathbf{G}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{Ga} \wedge \mathbf{Gb}$.

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Antimetric

$\mathbb{G}$

- Is an antiexomorphism such that $\mathbb{G}(\mathbf{a} \vee \mathbf{b}) = \mathbb{G}\mathbf{a} \vee \mathbb{G}\mathbf{b}$.

Metric and Antimetric

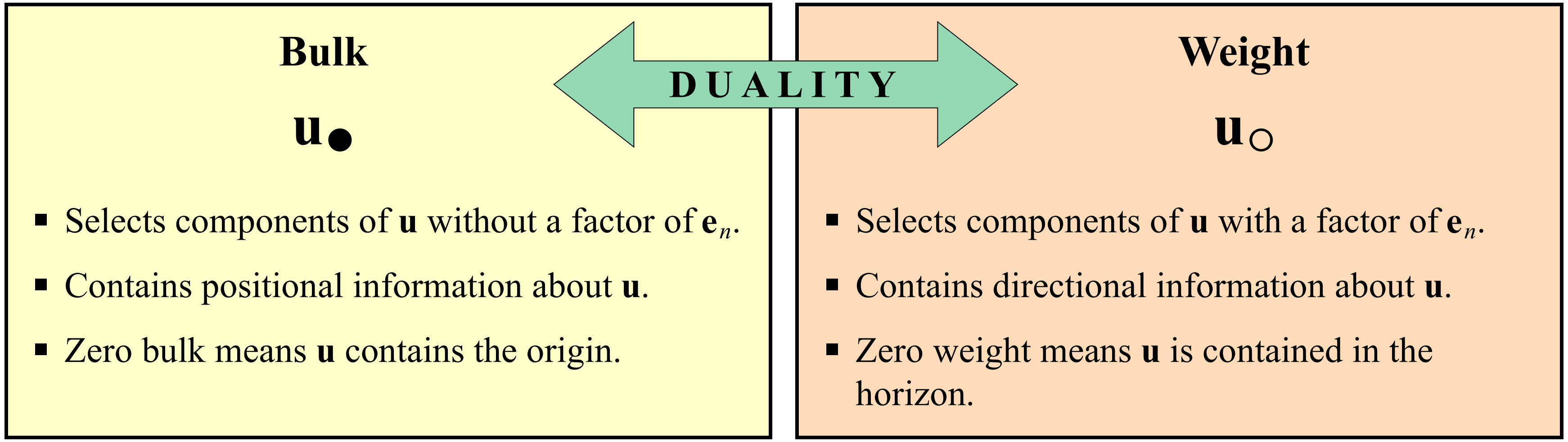
$$\mathbf{G} = \begin{bmatrix}
1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0
\end{bmatrix}
\qquad
\mathbb{G} = \begin{bmatrix}
0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1
\end{bmatrix}$$

$$\mathbf{G}\mathbb{G} = \det(\mathfrak{g})\,\mathbf{I}$$

Bulk and Weight

- Bulk $\mathbf{u}_{\bullet} = \mathbf{G}\mathbf{u}$ All components without factor $\mathbf{e}_4$

- Weight $\mathbf{u}_{\circ} = \mathbb{G}\mathbf{u}$ All components with factor $\mathbf{e}_4$



Inner Products

- Dot product defined by metric: $\mathbf{a} \bullet \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}} \mathbf{G} \mathbf{b}\right) \mathbf{1}$
- Antidot product defined by antimetric: $\mathbf{a} \circ \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b}\right) \mathbb{1}$
- Satisfies De Morgan law: $\mathbf{a} \circ \mathbf{b} = \overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}}$

Dot Product

$\mathbf{a} \bullet \mathbf{b}$

- Defined as $\left(\mathbf{a}^{\mathrm{T}} \mathbf{G} \mathbf{b}\right) \mathbf{1}$.
- Inner product between bulks.

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Antidot Product

$\mathbf{a} \circ \mathbf{b}$

- Defined as $\left(\mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b}\right) \mathbb{1}$.
- Inner product between weights.

Bulk and Weight Norms

- Two dot products induce two norms
- Bulk norm: $\left\| \mathbf{u} \right\|_{\bullet} = \sqrt{\mathbf{u} \bullet \mathbf{u}}$
- Weight norm: $\left\| \mathbf{u} \right\|_{\circ} = \sqrt{\mathbf{u} \circ \mathbf{u}}$
- Can generally have arbitrary values for same geometry due to homogeneity

Bulk and Weight Norms

Type	Bulk Norm	Weight Norm
Point $\mathbf{p}$	$\left\Vert \mathbf{p} \right\Vert_{\bullet} = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2}$	$\left\Vert \mathbf{p} \right\Vert_{\circ} = \lvert p_w \rvert \, \mathbb{1}$
Line $\boldsymbol{l}$	$\left\Vert \boldsymbol{l} \right\Vert_{\bullet} = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$	$\left\Vert \boldsymbol{l} \right\Vert_{\circ} = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$
Plane $\mathbf{g}$	$\left\Vert \mathbf{g} \right\Vert_{\bullet} = \lvert g_w \rvert \, \mathbf{1}$	$\left\Vert \mathbf{g} \right\Vert_{\circ} = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$

Unitization

- An object is *unitized* when its weight has magnitude one

Type	**Definition**	**Unitization**
Point $\mathbf{p}$	$\mathbf{p} = p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3 + p_w\,\mathbf{e}_4$	$p_w^2 = 1$
Line $\boldsymbol{l}$	$\boldsymbol{l} = l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43} + l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$	$l_{vx}^2 + l_{vy}^2 + l_{vz}^2 = 1$
Plane $\mathbf{g}$	$\mathbf{g} = g_x\,\mathbf{e}_{423} + g_y\,\mathbf{e}_{431} + g_z\,\mathbf{e}_{412} + g_w\,\mathbf{e}_{321}$	$g_x^2 + g_y^2 + g_z^2 = 1$

Geometric Norm

- Bulk and weight norms by themselves not very meaningful
- But add them, and result is a *homogeneous magnitude*
- Represents distance from origin
- Called the *geometric norm*

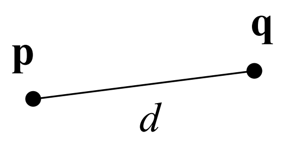
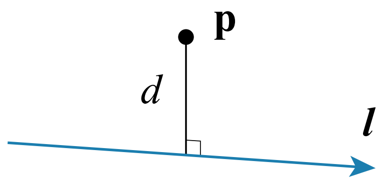
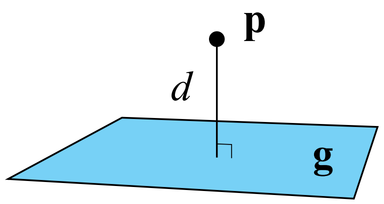
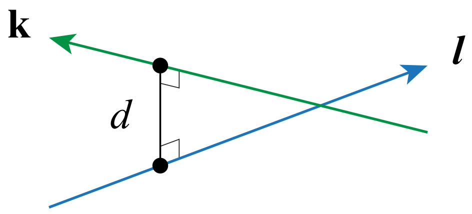
$$\|\mathbf{u}\| = \|\mathbf{u}\|_{\bullet} + \|\mathbf{u}\|_{\circ} = \sqrt{\mathbf{u} \bullet \mathbf{u}} + \sqrt{\mathbf{u} \circ \mathbf{u}}$$

- Two-component quantity, sum of scalar and antiscalar $s\mathbf{1} + t\mathbb{1}$
- Can be unitized by making weight one

Geometric Norm

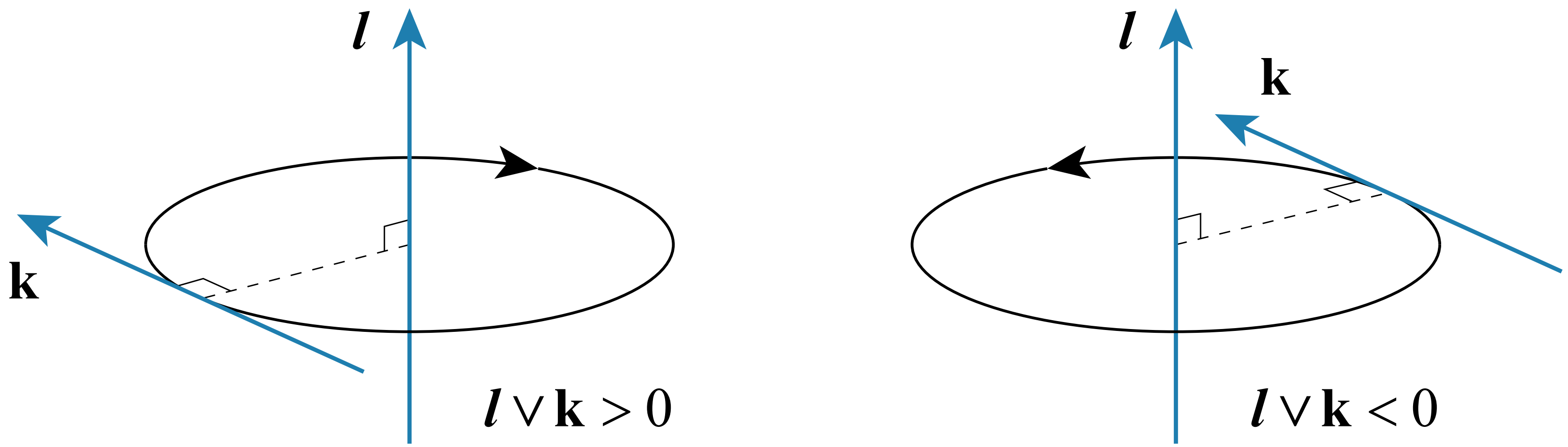
Type	Geometric Norm	Interpretation
Point **p**	$\left\|\widehat{\mathbf{p}}\right\| = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from the origin to the point **p**.
Line ***l***	$\left\|\widehat{\boldsymbol{l}}\right\| = \dfrac{\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}}{\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$	Perpendicular distance from the origin to the line ***l***.
Plane **g**	$\left\|\widehat{\mathbf{g}}\right\| = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$	Perpendicular distance from the origin to the plane **g**.

Euclidean Distance

Distance Formula	Illustration
Distance d between points $\mathbf{p}$ and $\mathbf{q}$. $d(\mathbf{p},\mathbf{q}) = \lVert \mathbf{q}_{xyz} p_w - \mathbf{p}_{xyz} q_w \rVert \mathbf{1} + \lvert p_w q_w \rvert \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and line $\boldsymbol{l}$. $d(\mathbf{p},\boldsymbol{l}) = \lVert \boldsymbol{l}_{\mathbf{v}} \times \mathbf{p}_{xyz} + p_w \boldsymbol{l}_{\mathbf{m}} \rVert \mathbf{1} + \lVert p_w \boldsymbol{l}_{\mathbf{v}} \rVert \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and plane $\mathbf{g}$. $d(\mathbf{p},\mathbf{g}) = (\mathbf{p} \cdot \mathbf{g}) \mathbf{1} + \lVert p_w \mathbf{g}_{xyz} \rVert \mathbb{1}$	
Perpendicular distance d between skew lines $\boldsymbol{l}$ and $\mathbf{k}$. $d(\boldsymbol{l},\mathbf{k}) = -(\boldsymbol{l}_{\mathbf{v}} \cdot \mathbf{k}_{\mathbf{m}} + \boldsymbol{l}_{\mathbf{m}} \cdot \mathbf{k}_{\mathbf{v}}) \mathbf{1} + \lVert \boldsymbol{l}_{\mathbf{v}} \times \mathbf{k}_{\mathbf{v}} \rVert \mathbb{1}$	

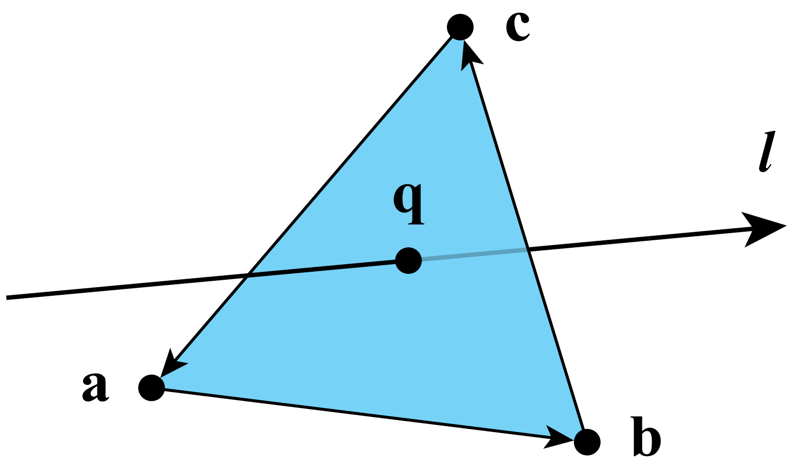
Line Crossing

- Sign of wedge product between lines gives crossing orientation

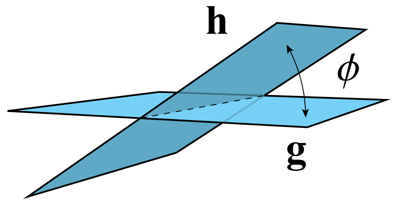
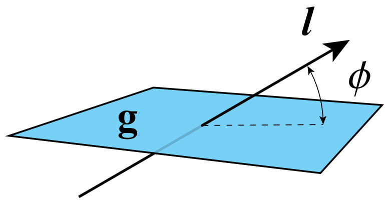
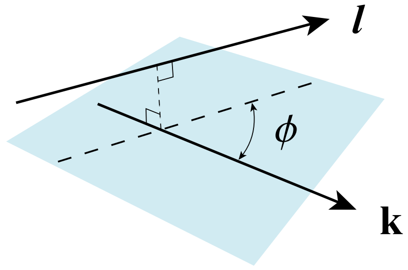


Line-Triangle Intersection

- Wedge product with all three edges of CCW-wound triangle must be positive



Euclidean Angle

Angle Formula	Illustration
Cosine of angle ϕ between planes **g** and **h**. $\cos\phi(\mathbf{g},\mathbf{h}) = (\mathbf{g}_{xyz}\cdot\mathbf{h}_{xyz})\,\mathbf{1}+\lVert\mathbf{g}\rVert_{\circ}\lVert\mathbf{h}\rVert_{\circ}$	
Cosine of angle ϕ between plane **g** and line $\boldsymbol{l}$. $\cos\phi(\mathbf{g},\boldsymbol{l}) = \lVert\mathbf{g}_{xyz}\times\boldsymbol{l}_{\mathbf{v}}\rVert\,\mathbf{1}+\lVert\mathbf{g}\rVert_{\circ}\lVert\boldsymbol{l}\rVert_{\circ}$	
Cosine of angle ϕ between lines $\boldsymbol{l}$ and **k**. $\cos\phi(\boldsymbol{l},\mathbf{k}) = (\boldsymbol{l}_{\mathbf{v}}\cdot\mathbf{k}_{\mathbf{v}})\,\mathbf{1}+\lVert\boldsymbol{l}\rVert_{\circ}\lVert\mathbf{k}\rVert_{\circ}$	

Bulk and Weight Duals

- Multiply by metric or antimetric, then take complement

Bulk Dual

$\mathbf{u}^{\bigstar}$

- Defined as $\overline{\mathbf{Gu}}$.
- Gives the dual of the bulk components of $\mathbf{u}$.

DUALITY

Weight Dual

$\mathbf{u}^{\medwhitestar}$

- Defined as $\overline{\mathbb{G}\mathbf{u}}$.
- Gives the dual of the weight components of $\mathbf{u}$.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{u}^{\bigstar}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0
$\mathbf{u}_{\bigstar}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$\mathbf{e}_4$	0
$\mathbf{u}^{\medwhitestar}$	0	0	0	0	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	0	$\mathbf{1}$
$\mathbf{u}_{\medwhitestar}$	0	0	0	0	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	$\mathbf{1}$

Interior Products

- Two exterior products combined with two duals
- Eight *interior* products using right and left duals

- Bulk contraction $\mathbf{a} \vee \mathbf{b}^{\bigstar}$ $\mathbf{b}_{\bigstar} \vee \mathbf{a}$
- Weight contraction $\mathbf{a} \vee \mathbf{b}^{\star}$ $\mathbf{b}_{\star} \vee \mathbf{a}$
- Bulk expansion $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$ $\mathbf{b}_{\bigstar} \wedge \mathbf{a}$
- Weight expansion $\mathbf{a} \wedge \mathbf{b}^{\star}$ $\mathbf{b}_{\star} \wedge \mathbf{a}$

Interior Products

- Right and left interior products differ by grade-dependent sign:

$$\mathbf{b}_{*} \vee \mathbf{a} = (-1)^{\mathrm{gr}(\mathbf{b})[\mathrm{gr}(\mathbf{a})+\mathrm{gr}(\mathbf{b})]}\, \mathbf{a} \vee \mathbf{b}^{*}$$

$$\mathbf{b}_{*} \wedge \mathbf{a} = (-1)^{\mathrm{ag}(\mathbf{b})[\mathrm{ag}(\mathbf{a})+\mathrm{ag}(\mathbf{b})]}\, \mathbf{a} \wedge \mathbf{b}^{*}$$

- Here, $*$ is either ★ or ☆

- Really need only four interior products

Interior Products

- Interior products reduce to inner products for same grade:

$$\mathbf{a} \vee \mathbf{b}^{\bigstar} = \mathbf{a} \bullet \mathbf{b}, \qquad \text{when } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b})$$

$$\mathbf{a} \vee \mathbf{b}^{\star} = (\mathbf{a} \circ \mathbf{b}) \vee \mathbf{1}, \qquad \text{when } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b})$$

$$\mathbf{a} \wedge \mathbf{b}^{\star} = \mathbf{a} \circ \mathbf{b}, \qquad \text{when } \operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b})$$

$$\mathbf{a} \wedge \mathbf{b}^{\bigstar} = (\mathbf{a} \bullet \mathbf{b}) \wedge \mathbb{1}, \qquad \text{when } \operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b})$$

Contraction and Expansion

- Subtract grades or antigrades

Contraction

$\mathbf{a} \vee \mathbf{b}^{\bigstar}$

- Decreases the grade of **a** by removing the subspace spanned by **b**.
- The result is a lower-dimensional object that is contained by **a** and is orthogonal to **b**.
- The grade of the result is $\operatorname{gr}(\mathbf{a}) - \operatorname{gr}(\mathbf{b})$.

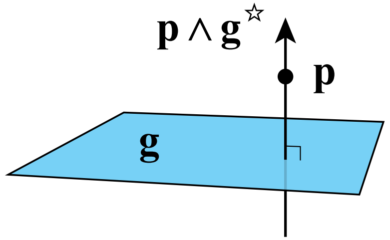
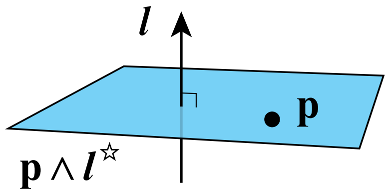
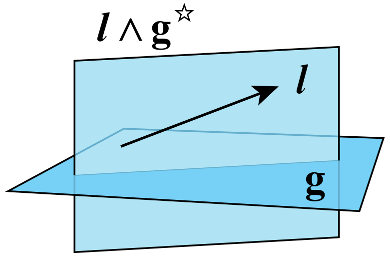
DUALITY

Expansion

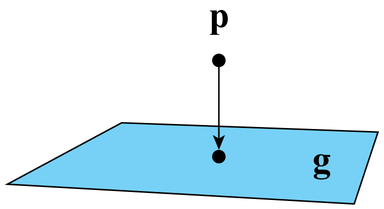
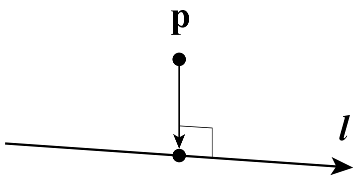
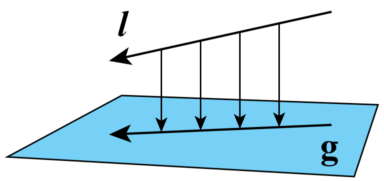
$\mathbf{a} \wedge \mathbf{b}^{\star}$

- Increases the grade of **a** by adding the subspace not spanned by **b**.
- The result is a higher-dimensional object that contains **a** and is orthogonal to **b**.
- The antigrade of the result is $\operatorname{ag}(\mathbf{a}) - \operatorname{ag}(\mathbf{b})$.

Weight Expansion

Expansion Operation	Illustration
Line containing point **p** and orthogonal to plane **g**. $$\mathbf{p} \wedge \mathbf{g}^{\star} = -p_w g_x \mathbf{e}_{41} - p_w g_y \mathbf{e}_{42} - p_w g_z \mathbf{e}_{43} + (p_z g_y - p_y g_z)\mathbf{e}_{23} + (p_x g_z - p_z g_x)\mathbf{e}_{31} + (p_y g_x - p_x g_y)\mathbf{e}_{12}$$	
Plane containing point **p** and orthogonal to line ***l***. $$\mathbf{p} \wedge \boldsymbol{l}^{\star} = -p_w l_{vx} \mathbf{e}_{423} - p_w l_{vy} \mathbf{e}_{431} - p_w l_{vz} \mathbf{e}_{412} + (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\mathbf{e}_{321}$$	
Plane containing line ***l*** and orthogonal to plane **g**. $$\boldsymbol{l} \wedge \mathbf{g}^{\star} = (l_{vy} g_z - l_{vz} g_y)\mathbf{e}_{423} + (l_{vz} g_x - l_{vx} g_z)\mathbf{e}_{431} + (l_{vx} g_y - l_{vy} g_x)\mathbf{e}_{412} - (l_{mx} g_x + l_{my} g_y + l_{mz} g_z)\mathbf{e}_{321}$$	

Orthogonal Projection

Projection Operation	Illustration
Orthogonal projection of point **p** onto plane **g**. $\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{\star}) = (g_x^2 + g_y^2 + g_z^2)(p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4) - (g_x p_x + g_y p_y + g_z p_z + g_w p_w)(g_x\mathbf{e}_1 + g_y\mathbf{e}_2 + g_z\mathbf{e}_3)$	
Orthogonal projection of point **p** onto line ***l***. $\boldsymbol{l} \vee (\mathbf{p} \wedge \boldsymbol{l}^{\star}) = (l_{vx} p_x + l_{vy} p_y + l_{vz} p_z)(l_{vx}\mathbf{e}_1 + l_{vy}\mathbf{e}_2 + l_{vz}\mathbf{e}_3) + (l_{vx}^2 + l_{vy}^2 + l_{vz}^2)\, p_w\mathbf{e}_4 + (l_{vy} l_{mz} - l_{vz} l_{my})\, p_w\mathbf{e}_1 + (l_{vz} l_{mx} - l_{vx} l_{mz})\, p_w\mathbf{e}_2 + (l_{vx} l_{my} - l_{vy} l_{mx})\, p_w\mathbf{e}_3$	
Orthogonal projection of line ***l*** onto plane **g**. $\mathbf{g} \vee (\boldsymbol{l} \wedge \mathbf{g}^{\star}) = (g_x^2 + g_y^2 + g_z^2)(l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43}) - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})(g_x\mathbf{e}_{41} + g_y\mathbf{e}_{42} + g_z\mathbf{e}_{43}) + (g_x l_{mx} + g_y l_{my} + g_z l_{mz})(g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31} + g_z\mathbf{e}_{12}) + (g_z l_{vy} - g_y l_{vz})\, g_w\mathbf{e}_{23} + (g_x l_{vz} - g_z l_{vx})\, g_w\mathbf{e}_{31} + (g_y l_{vx} - g_x l_{vy})\, g_w\mathbf{e}_{12}$	

Support

- Orthogonal projection of origin onto line or plane
- Support is point closest to origin contained by object

$$\sup(\boldsymbol{l}) = (l_{vy}l_{mz} - l_{vz}l_{my})\mathbf{e}_1 + (l_{vz}l_{mx} - l_{vx}l_{mz})\mathbf{e}_2 + (l_{vx}l_{my} - l_{vy}l_{mx})\mathbf{e}_3 + \boldsymbol{l}_{\mathbf{v}}^2\mathbf{e}_4$$

$$\sup(\mathbf{g}) = -g_x g_w \mathbf{e}_1 - g_y g_w \mathbf{e}_2 - g_z g_w \mathbf{e}_3 + \left(g_x^2 + g_y^2 + g_z^2\right)\mathbf{e}_4$$

Geometric / Clifford Algebra

- Geometric product $\mathbf{a} ⟑ \mathbf{b}$

- Geometric antiproduct $\mathbf{a} ⟇ \mathbf{b}$

- We use upward and downward wedge with dot inside
- “Wedge-dot” and “Antiwedge-dot”

- G.P. historically denoted by juxtaposition without symbol
- But duality gives us two products that need distinguishing

Geometric Product and Antiproduct

- Vectors square to inner product instead of zero
- Product satisfy the usual De Morgan law

$$\mathbf{a} ⟇ \mathbf{b} = \overline{\underline{\mathbf{a}} ⟑ \underline{\mathbf{b}}}$$

Geometric Product

$\mathbf{a} ⟑ \mathbf{b}$

- Vector **v** squares to dot product $\mathbf{v} \bullet \mathbf{v}$.
- Includes wedge product $\wedge$.
- Identity is scalar **1**.

DUALITY

Geometric Antiproduct

$\mathbf{a} ⟇ \mathbf{b}$

- Antivector squares to antidot product $\mathbf{v} \circ \mathbf{v}$.
- Includes antiwedge product $\vee$.
- Identity is antiscalar $\mathbb{1}$.

Geometric Products

- For <u>vectors</u> **a** and **b**:

$$\mathbf{a} \mathbin{\dot\wedge} \mathbf{b} = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

- For <u>antivectors</u> **a** and **b**:

$$\mathbf{a} \mathbin{\dot\vee} \mathbf{b} = \mathbf{a} \circ \mathbf{b} + \mathbf{a} \vee \mathbf{b}$$

4D Geometric Product

Geometric Product $\mathbf{a} ⟑ \mathbf{b}$

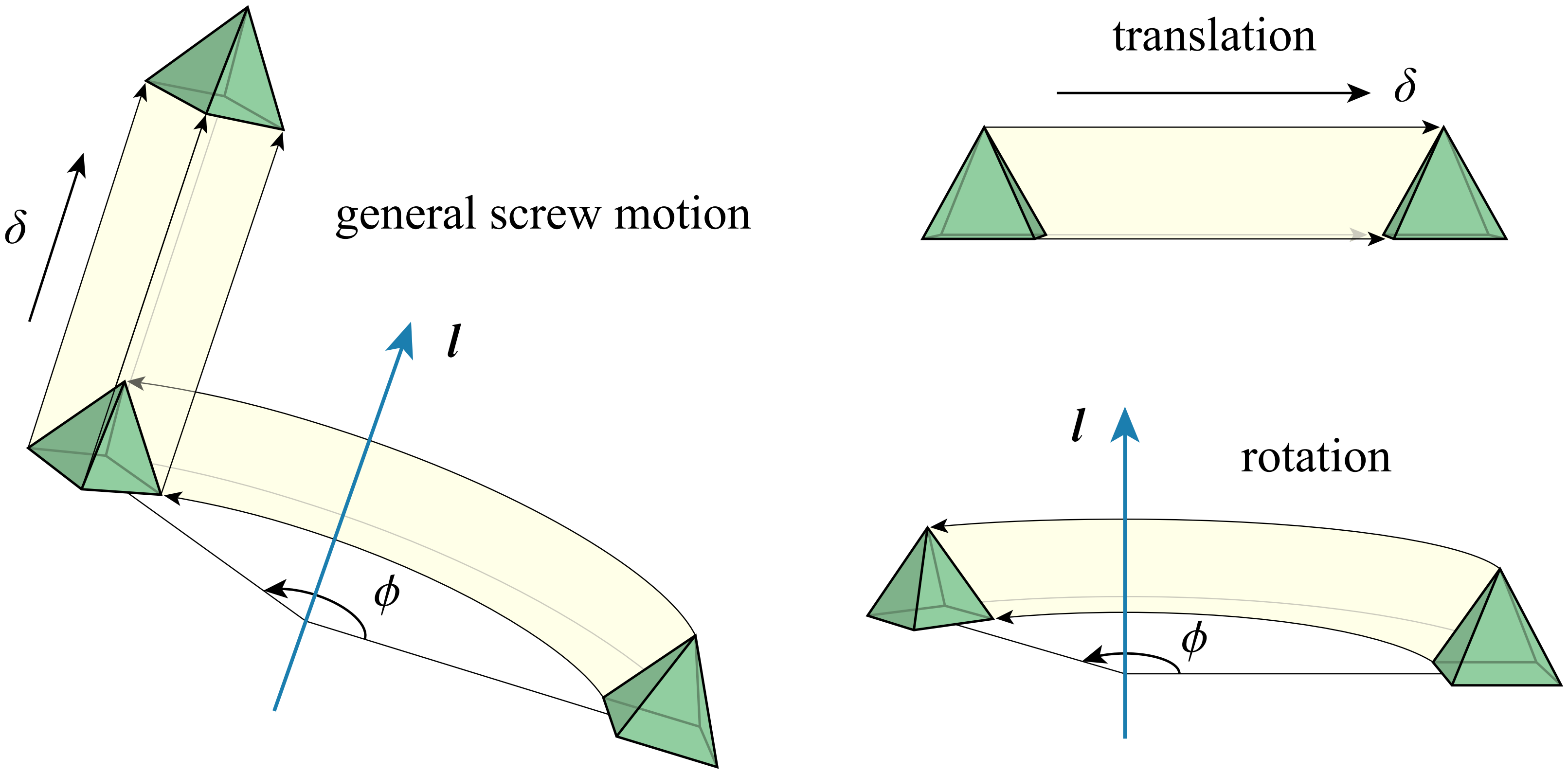
a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	$\mathbf{e}_3$	$-\mathbf{e}_{321}$	$-\mathbf{e}_1$	$-\mathbf{e}_{43}$	$\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{1}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbb{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_4$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	$-\mathbf{e}_{423}$	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$\mathbf{e}_4$	$\mathbf{e}_{423}$	0	0	0	0	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	$-\mathbf{e}_{431}$	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbf{e}_4$	0	0	0	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	$-\mathbf{e}_{412}$	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$-\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$-\mathbf{e}_{321}$	$-\mathbf{e}_1$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	$-\mathbf{1}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	0	0	0	$\mathbf{e}_{41}$	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	0	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	0	0	0	$\mathbf{e}_{42}$	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	0	0	0	$\mathbf{e}_{43}$	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{1}$	$\mathbf{e}_4$
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0

4D Geometric Antiproduct

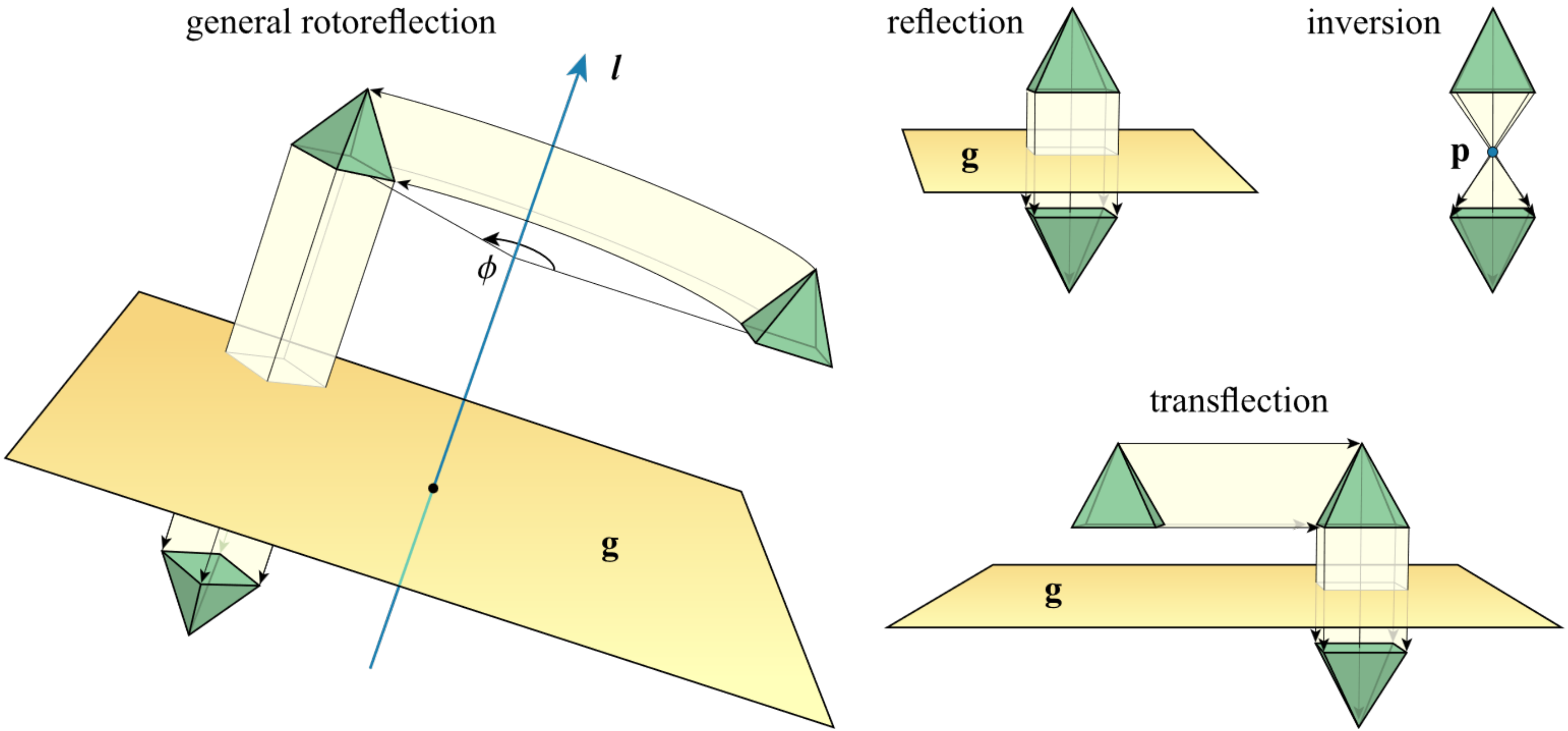
Geometric Antiproduct $\mathbf{a} \mathbin{\vee\!\!\!\!\!\wedge} \mathbf{b}$

$\mathbf{a}$ \ $\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	0	$\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	0	$\mathbf{1}$
$\mathbf{e}_{1}$	0	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	0	0	0	$\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	0	$\mathbf{e}_{1}$
$\mathbf{e}_{2}$	0	0	0	0	$-\mathbf{e}_{31}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	0	0	0	$\mathbf{e}_{12}$	$\mathbf{1}$	$-\mathbf{e}_{23}$	0	$\mathbf{e}_{2}$
$\mathbf{e}_{3}$	0	0	0	0	$-\mathbf{e}_{12}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	0	0	0	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{1}$	0	$\mathbf{e}_{3}$
$\mathbf{e}_{4}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{2}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{1}$	$\mathbf{e}_{4}$
$\mathbf{e}_{41}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{1}$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{43}$	$-\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423}$	$\mathbf{e}_{2}$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	$\mathbf{e}_{12}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbb{1}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{3}$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	$\mathbf{e}_{1}$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	0	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	0	0	0	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	0	0	0	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	$-\mathbf{e}_{1}$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	$-\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423}$	$\mathbf{e}_{3}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$\mathbf{e}_{43}$	$\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbb{1}$	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$-\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Proper Euclidean Isometries



Improper Euclidean Isometries



Geometric Product

- Geometric **product** in 4D space fixes the origin
- Cannot perform transformations we want

- Geometric **antiproduct** performs Euclidean isometries
- Uses sandwiching similar to quaternions

Plane Reflection

- Sandwich antiproduct with plane **g** performs reflection:

$$\mathbf{u}' = \mathbf{g} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \mathbf{g}$$

- Multiple reflections stack outward from **u**:

$$\mathbf{u}' = \left(\mathbf{h} \mathbin{\dot{\vee}} \mathbf{g}\right) \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \left(\mathbf{g} \mathbin{\dot{\vee}} \mathbf{h}\right)$$

- Basis for all Euclidean isometries

Reverse and Antireverse

- Multiply vector or antivector factors in reverse order

Reverse

$\tilde{\mathbf{u}}$

- Reverses the order in which the vector factors of **u** are multiplied under the wedge product.
- Changes sign when the grade of **u** is 2 or 3 modulo 4.

DUALITY

Antireverse

$\underset{\sim}{\mathbf{u}}$

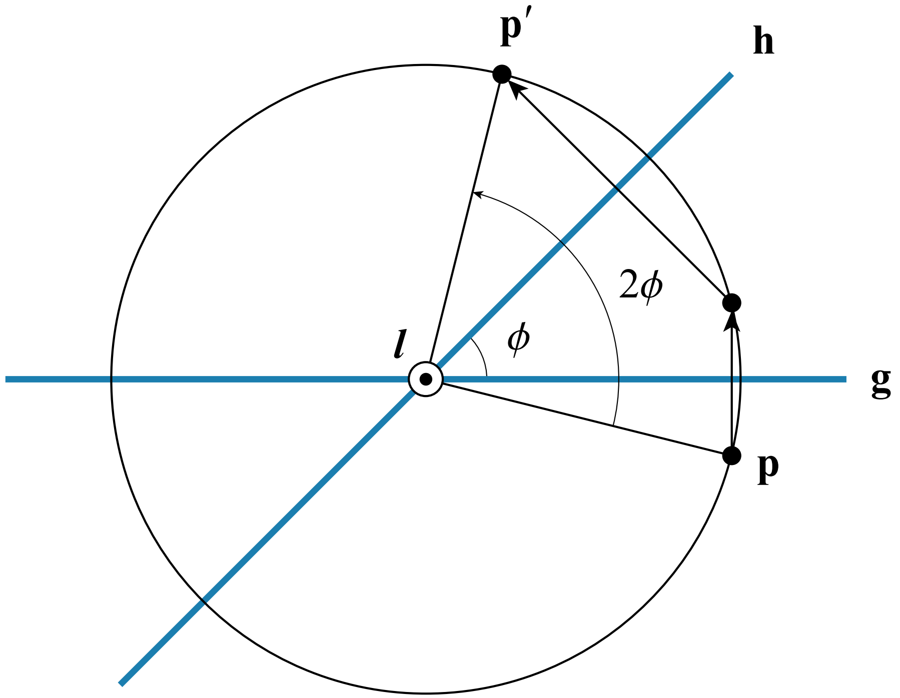
- Reverses the order in which the antivector factors of **u** are multiplied under the antiwedge product.
- Changes sign when the antigrade of **u** is 2 or 3 modulo 4.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\tilde{\mathbf{u}}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$\mathbb{1}$
$\underset{\sim}{\mathbf{u}}$	$\mathbf{1}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Rotation about a Line

- Let **g** and **h** be planes meeting at an angle ϕ
- Reflection across **g** followed by **h** is rotation through 2ϕ about line ***l*** where planes intersect

$$l = \frac{\mathbf{h} \vee \mathbf{g}}{\left\| \mathbf{h} \vee \mathbf{g} \right\|_{\circ}}$$



Rotation about a Line

- Planes multiply together under geometric antiproduct to form rotation operator **R**

$$\mathbf{p}' = \mathbf{h} \mathbin{\dot{\vee}} \left(\mathbf{g} \mathbin{\dot{\vee}} \mathbf{p} \mathbin{\dot{\vee}} \mathbf{g} \right) \mathbin{\dot{\vee}} \mathbf{h}$$

$$\mathbf{p}' = \mathbf{R} \mathbin{\dot{\vee}} \mathbf{p} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{R}}$$

$$\mathbf{R} = \mathbf{h} \mathbin{\dot{\vee}} \mathbf{g}$$

Rotation about a Line

- General form of rotation operator **R**:

$$\mathbf{R} = \boldsymbol{l} \sin \phi + \mathbb{1} \cos \phi$$

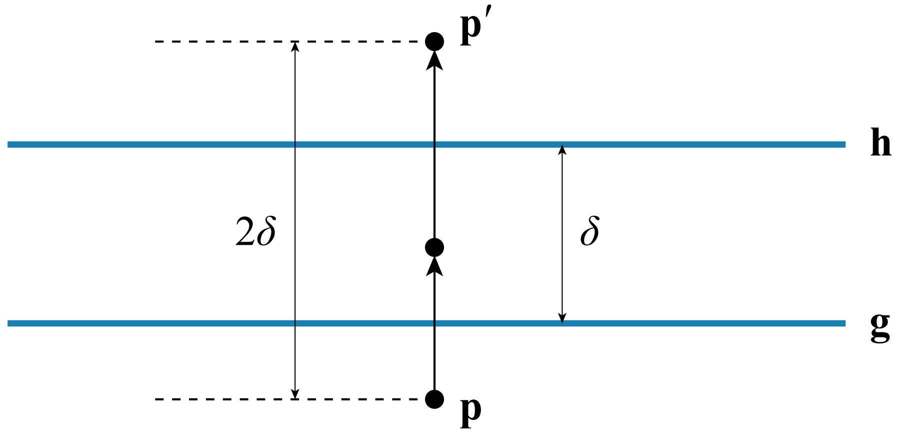
- Rotates through angle 2ϕ about unitized line ***l***

$$\mathbf{u}' = \mathbf{R} \mathbin{\dot\vee} \mathbf{u} \mathbin{\dot\vee} \underset{\sim}{\mathbf{R}}$$

- Rotates any geometry and even other operators

Translation

- If planes **g** and **h** are parallel, result is a translation
- Translation goes along normal direction by twice the distance δ between the planes



Translation

- General form of translation operator **T**:

$$\mathbf{T} = \tau_x \mathbf{e}_{23} + \tau_y \mathbf{e}_{31} + \tau_z \mathbf{e}_{12} + \mathbb{1}$$

- Translates by displacement vector 2**τ**

$$\mathbf{u}' = \mathbf{T} \mathbin{\dot\vee} \mathbf{u} \mathbin{\dot\vee} \underset{\sim}{\mathbf{T}}$$

- Translates any geometry and even other operators

Euclidean Isometry Operators

- Sandwiches with geometric antiproduct perform Euclidean isometries
- Motor = MOtion operaTOR
- Flector = reFLECtion operaTOR

Motor

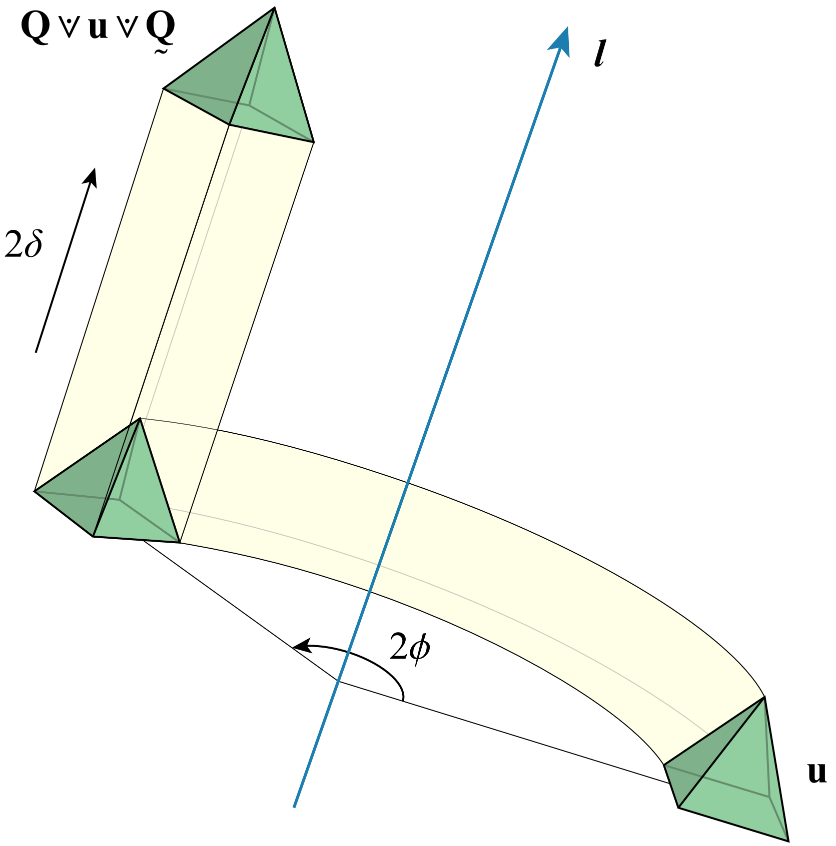
- General form of a motor:

$$\mathbf{Q} = \underbrace{Q_{vx}\mathbf{e}_{41} + Q_{vy}\mathbf{e}_{42} + Q_{vz}\mathbf{e}_{43} + Q_{vw}\mathbb{1}}_{\text{Rotation Quaternion}} + \underbrace{Q_{mx}\mathbf{e}_{23} + Q_{my}\mathbf{e}_{31} + Q_{mz}\mathbf{e}_{12} + Q_{mw}\mathbf{1}}_{\text{Moment and Displacement}}$$

- Performs any combination of rotations and translations

$$\mathbf{u}' = \mathbf{Q} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{Q}}$$

Motor



$$\mathbf{Q} = \exp_{\curlyvee}\left[\left(\delta\mathbf{1} + \varphi\mathbb{1}\right)\curlyvee \boldsymbol{l}\right] = \boldsymbol{l}\sin\varphi - \boldsymbol{l}^{\star}\delta\cos\varphi - \delta\sin\varphi + \mathbb{1}\cos\varphi$$

Flector

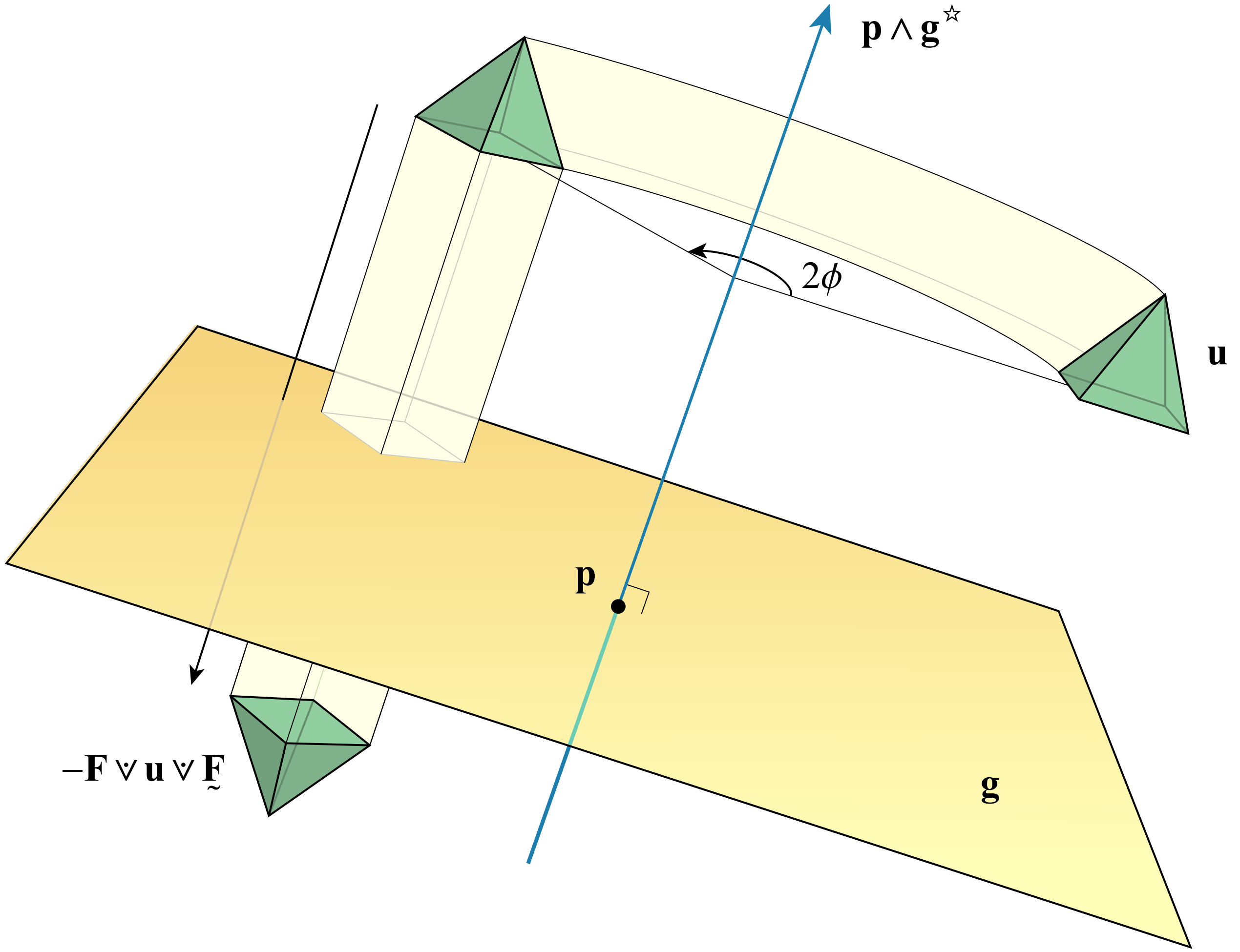
- General form of a flector:

$$\mathbf{F} = F_{px}\,\mathbf{e}_1 + F_{py}\,\mathbf{e}_2 + F_{pz}\,\mathbf{e}_3 + F_{pw}\,\mathbf{e}_4 + F_{gx}\,\mathbf{e}_{423} + F_{gy}\,\mathbf{e}_{431} + F_{gz}\,\mathbf{e}_{412} + F_{gw}\,\mathbf{e}_{321}$$

Point (first four terms) — Plane (last four terms)

- Performs any combination of rotoreflections

Flector



$$\mathbf{F} = \mathbf{p}\sin\phi + \mathbf{g}\cos\phi$$

Motor Parameterization

- A motion operator is parameterized by:
  - A unitized line ***l***
  - A rotation angle ϕ
  - A displacement distance δ

- Exponential with respect to geometric antiproduct:

$$\mathbf{Q} = \exp_{\veebar}\left[\left(\delta\mathbf{1} + \phi\mathbb{1}\right) \veebar \boldsymbol{l}\right] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

- $\delta\mathbf{1} + \phi\mathbb{1}$ is *pitch* of screw transformation

Motor Parameterization

- Given arbitrary motor **Q**, can calculate parameters

$$\mathbf{Q} = Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1} + Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}$$

$$\mathbf{Q} = \exp_{\veebar}\left[(\delta\mathbf{1} + \phi\mathbb{1}) \veebar \boldsymbol{l}\right] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

$$s = \sin\phi = \sqrt{1 - Q_{vw}^2} \qquad \delta = -\frac{Q_{mw}}{s} \qquad \phi = \tan^{-1}\left(\frac{s}{Q_{vw}}\right)$$

$$\boldsymbol{l}_{\mathbf{v}} = \frac{1}{s}\mathbf{Q}_{vxyz} \qquad \boldsymbol{l}_{\mathbf{m}} = \frac{1}{s}\left(\mathbf{Q}_{mxyz} + \frac{Q_{vw}Q_{mw}}{s^2}\mathbf{Q}_{vxyz}\right)$$

Motor Interpolation

- To interpolate from motor $\mathbf{Q}_1$ to motor $\mathbf{Q}_2$, first calculate

$$\mathbf{Q}_0 = \mathbf{Q}_2 \mathbin{\dot\vee} \mathbf{Q}_1^{-1} = \mathbf{Q}_2 \mathbin{\dot\vee} \underset{\sim}{\mathbf{Q}_1}$$

- Then calculate parameters $\boldsymbol{l}$, δ, and ϕ for $\mathbf{Q}_0$

- Interpolate from identity $\mathbb{1}$ to $\mathbf{Q}_0$ with

$$\mathbf{Q}(t) = \exp_{\vee}\left[t\left(\delta\mathbf{1} + \phi\mathbb{1}\right)\vee\boldsymbol{l}\right] = \boldsymbol{l}\sin(t\phi) - \boldsymbol{l}^{\star} t\delta\cos(t\phi) - t\delta\sin(t\phi) + \mathbb{1}\cos(t\phi)$$

- Finally, calculate $\mathbf{Q}(t) \mathbin{\dot\vee} \mathbf{Q}_1$

Motor Interpolation

- That can be computationally expensive

- Approximate interpolation is often acceptable:

$$\mathbf{Q}(t) = (1-t)\,\mathbf{Q}_1 + t\mathbf{Q}_2$$

- This needs to be unitized and constrained

$$\frac{\mathbf{Q}}{\left\|\mathbf{Q_v}\right\|} \,\dot{\vee} \left(-\frac{\mathbf{Q_v} \cdot \mathbf{Q_m}}{\mathbf{Q_v^2}} \mathbf{1} + \mathbb{1} \right) = \frac{1}{\left\|\mathbf{Q_v}\right\|} \left[\mathbf{Q} - \frac{\mathbf{Q_v} \cdot \mathbf{Q_m}}{\mathbf{Q_v^2}} \left(Q_{vx}\,\mathbf{e}_{23} + Q_{vy}\,\mathbf{e}_{31} + Q_{vz}\,\mathbf{e}_{12} + Q_{vw} \right) \right]$$

Square Root of Motor

- Special case of interpolation from $\mathbb{1}$ to **Q** when *t* = 1/2

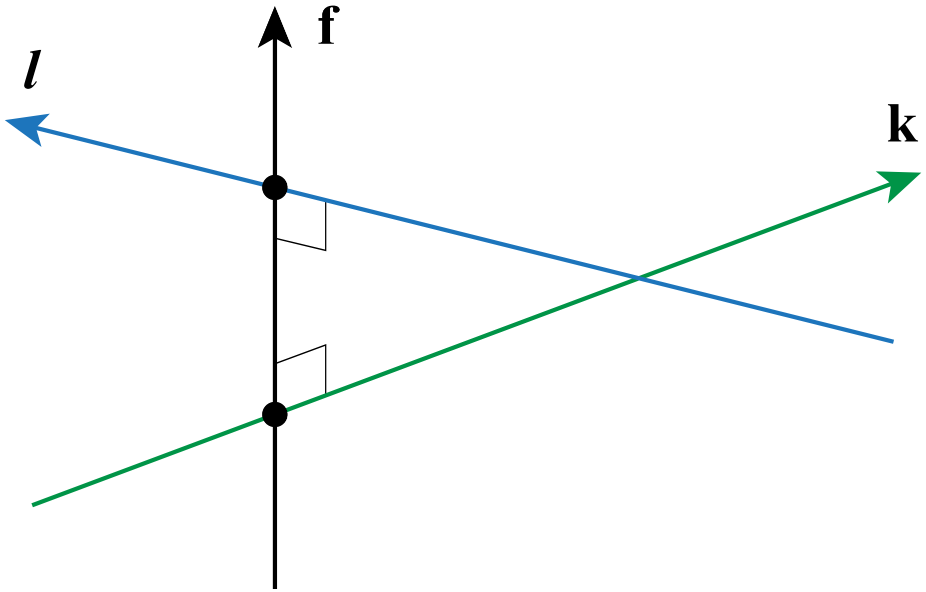
$$\sqrt[\dot{\vee}]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\sqrt{2 + 2Q_{\mathbb{1}}}} \mathbin{\dot{\vee}} \left(\mathbb{1} - \frac{Q_{\mathbf{1}}}{2 + 2Q_{\mathbb{1}}} \mathbf{1} \right)$$

- For simple motor (pure rotation or translation), this simplifies:

$$\sqrt[\dot{\vee}]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\left\| \mathbf{Q} + \mathbb{1} \right\|_{\circ}}$$

Line to Line Motion

- Let **k** and ***l*** be lines separated by distance δ with angle ϕ between directions
- Operator $\boldsymbol{l} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{k}}$ rotates by 2ϕ and translates by distance 2δ about line **f** connecting closest points
- Square root of this operator transforms line **k** into line ***l***



Motor-Point Transformation

- 25 multiply-adds:

$$\mathbf{p}'_{xyz} = \mathbf{p}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v}\times\mathbf{a} - Q_{mw}\,p_w\,\mathbf{v}\right)$$

$$p'_w = p_w$$

$$\mathbf{a} = \mathbf{v}\times\mathbf{p}_{xyz} + p_w\mathbf{m}$$

$$\mathbf{v} = \left(Q_{vx}, Q_{vy}, Q_{vz}\right)$$

$$\mathbf{m} = \left(Q_{mx}, Q_{my}, Q_{mz}\right)$$

- 3×4 matrix transformation only requires 12 multiply-adds, (or just 9 if $p_w = 1$)

Motor-Line Transformation

- 54 multiply-adds:

$$\boldsymbol{l}'_{\mathbf{v}} = \boldsymbol{l}_{\mathbf{v}} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a}\right)$$

$$\boldsymbol{l}'_{\mathbf{m}} = \boldsymbol{l}_{\mathbf{m}} + 2\left[Q_{mw}\mathbf{a} + Q_{vw}\left(\mathbf{b} + \mathbf{c}\right) + \mathbf{v} \times \left(\mathbf{b} + \mathbf{c}\right) + \mathbf{m} \times \mathbf{a}\right]$$

$$\mathbf{a} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{v}} \qquad \mathbf{b} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{m}} \qquad \mathbf{c} = \mathbf{m} \times \boldsymbol{l}_{\mathbf{v}}$$

- 6×6 matrix transformation only requires 27 multiply-adds

Motor-Plane Transformation

- 35 multiply-adds:

$$\mathbf{g}'_{xyz} = \mathbf{g}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a}\right)$$

$$g'_w = g_w + 2\left[\left(\mathbf{m} \times \mathbf{g}_{xyz} + Q_{mw}\mathbf{g}_{xyz}\right) \cdot \mathbf{v} - Q_{vw}\left(\mathbf{m} \cdot \mathbf{g}_{xyz}\right)\right]$$

$$\mathbf{a} = \mathbf{v} \times \mathbf{g}_{xyz}$$

- 4×4 matrix transformation only requires 13 multiply-adds

Motor to Matrix

$$\mathbf{A_Q} = \begin{bmatrix} 1-2\left(Q_{vy}^2+Q_{vz}^2\right) & 2Q_{vx}Q_{vy} & 2Q_{vz}Q_{vx} & 2\left(Q_{vy}Q_{mz}-Q_{vz}Q_{my}\right) \\ 2Q_{vx}Q_{vy} & 1-2\left(Q_{vz}^2+Q_{vx}^2\right) & 2Q_{vy}Q_{vz} & 2\left(Q_{vz}Q_{mx}-Q_{vx}Q_{mz}\right) \\ 2Q_{vz}Q_{vx} & 2Q_{vy}Q_{vz} & 1-2\left(Q_{vx}^2+Q_{vy}^2\right) & 2\left(Q_{vx}Q_{my}-Q_{vy}Q_{mx}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B_Q} = \begin{bmatrix} 0 & -2Q_{vz}Q_{vw} & 2Q_{vy}Q_{vw} & 2\left(Q_{vw}Q_{mx}-Q_{vx}Q_{mw}\right) \\ 2Q_{vz}Q_{vw} & 0 & -2Q_{vx}Q_{vw} & 2\left(Q_{vw}Q_{my}-Q_{vy}Q_{mw}\right) \\ -2Q_{vy}Q_{vw} & 2Q_{vx}Q_{vw} & 0 & 2\left(Q_{vw}Q_{mz}-Q_{vz}Q_{mw}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M_Q} = \mathbf{A_Q} + \mathbf{B_Q} \qquad \mathbf{M_Q^{-1}} = \mathbf{A_Q} - \mathbf{B_Q}$$

Motor Composition

- 48 multiply-adds:

$$
\begin{aligned}
\mathbf{Q} \vee \mathbf{R} = &\left(Q_{vw}R_{vx} + Q_{vx}R_{vw} + Q_{vy}R_{vz} - Q_{vz}R_{vy}\right)\mathbf{e}_{41} \\
&+\left(Q_{vw}R_{vy} - Q_{vx}R_{vz} + Q_{vy}R_{vw} + Q_{vz}R_{vx}\right)\mathbf{e}_{42} \\
&+\left(Q_{vw}R_{vz} + Q_{vx}R_{vy} - Q_{vy}R_{vx} + Q_{vz}R_{vw}\right)\mathbf{e}_{43} \\
&+\left(Q_{vw}R_{vw} - Q_{vx}R_{vx} - Q_{vy}R_{vy} - Q_{vz}R_{vz}\right)\mathbb{1} \\
&+\left(Q_{mw}R_{vx} + Q_{mx}R_{vw} + Q_{my}R_{vz} - Q_{mz}R_{vy} + Q_{vw}R_{mx} + Q_{vx}R_{mw} + Q_{vy}R_{mz} - Q_{vz}R_{my}\right)\mathbf{e}_{23} \\
&+\left(Q_{mw}R_{vy} - Q_{mx}R_{vz} + Q_{my}R_{vw} + Q_{mz}R_{vx} + Q_{vw}R_{my} - Q_{vx}R_{mz} + Q_{vy}R_{mw} + Q_{vz}R_{mx}\right)\mathbf{e}_{31} \\
&+\left(Q_{mw}R_{vz} + Q_{mx}R_{vy} - Q_{my}R_{vx} + Q_{mz}R_{vw} + Q_{vw}R_{mz} + Q_{vx}R_{my} - Q_{vy}R_{mx} + Q_{vz}R_{mw}\right)\mathbf{e}_{12} \\
&+\left(Q_{mw}R_{vw} - Q_{mx}R_{vx} - Q_{my}R_{vy} - Q_{mz}R_{vz} + Q_{vw}R_{mw} - Q_{vx}R_{mx} - Q_{vy}R_{my} - Q_{vz}R_{mz}\right)\mathbf{1}
\end{aligned}
$$

- Composition of equivalent 3×4 matrices requires 33 multiply-adds

Motor and Matrix

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_x \\ m_{21} & m_{22} & m_{23} & t_y \\ m_{31} & m_{32} & m_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Q} = Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1} + Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}$$

$$\mathbf{M} = \begin{bmatrix} 1-2\left(Q_{vy}^2 + Q_{vz}^2\right) & 2\left(Q_{vx}Q_{vy} - Q_{vz}Q_{vw}\right) & 2\left(Q_{vz}Q_{vx} + Q_{vy}Q_{vw}\right) & 2\left(Q_{vy}Q_{mz} - Q_{vz}Q_{my} + Q_{vw}Q_{mx} - Q_{vx}Q_{mw}\right) \\ 2\left(Q_{vx}Q_{vy} + Q_{vz}Q_{vw}\right) & 1-2\left(Q_{vy}^2 + Q_{vz}^2\right) & 2\left(Q_{vy}Q_{vz} - Q_{vx}Q_{vw}\right) & 2\left(Q_{vz}Q_{mx} - Q_{vx}Q_{mz} + Q_{vw}Q_{my} - Q_{vy}Q_{mw}\right) \\ 2\left(Q_{vz}Q_{vx} - Q_{vy}Q_{vw}\right) & 2\left(Q_{vy}Q_{vz} + Q_{vx}Q_{vw}\right) & 1-2\left(Q_{vx}^2 + Q_{vy}^2\right) & 2\left(Q_{vx}Q_{my} - Q_{vy}Q_{mx} + Q_{vw}Q_{mz} - Q_{vz}Q_{mw}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

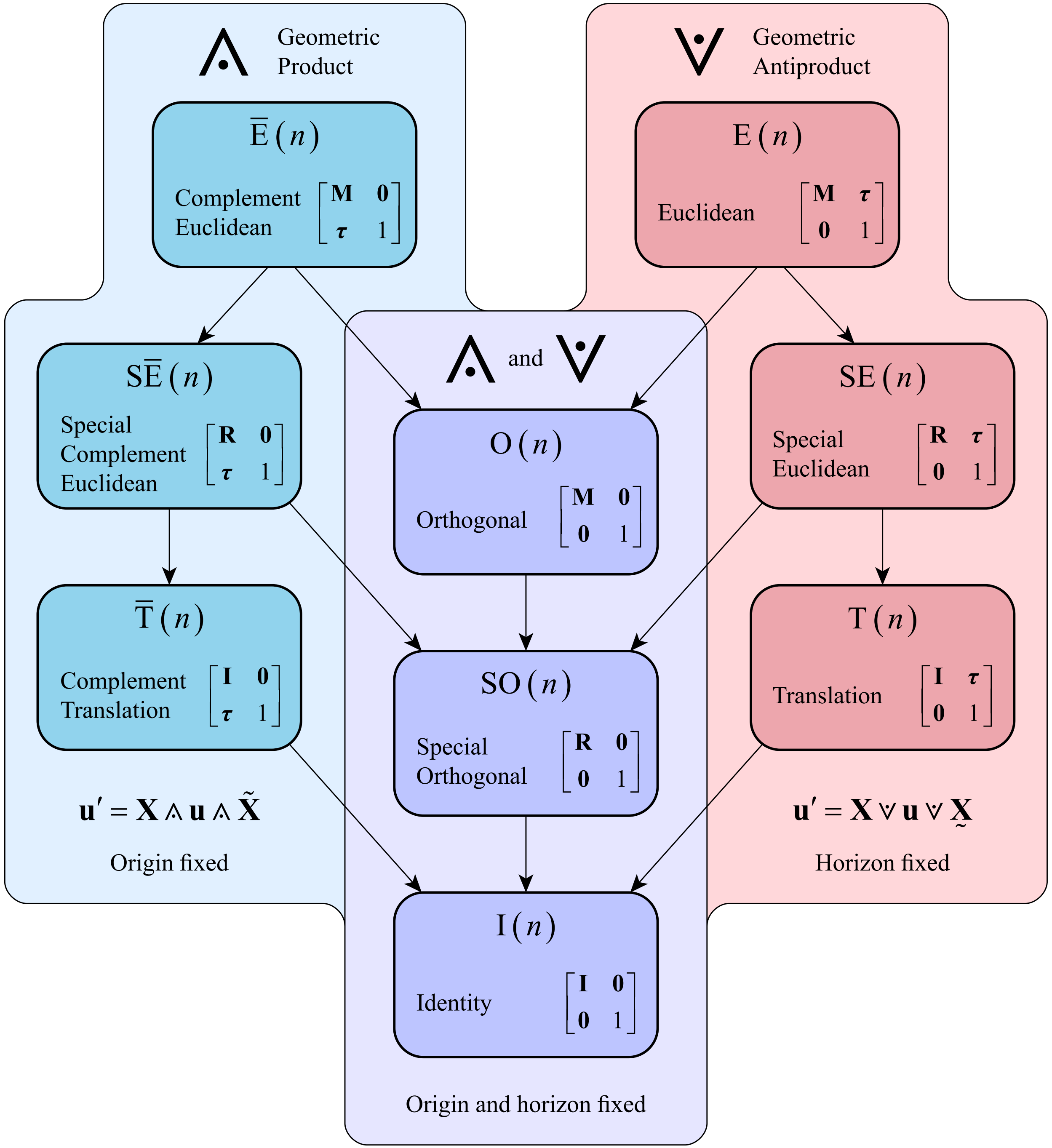
Matrix Advantages

- Can represent more transformations
- Can read off origin and axis directions in transformed space
- Faster to transform objects
- Faster to compose

Motor Advantages

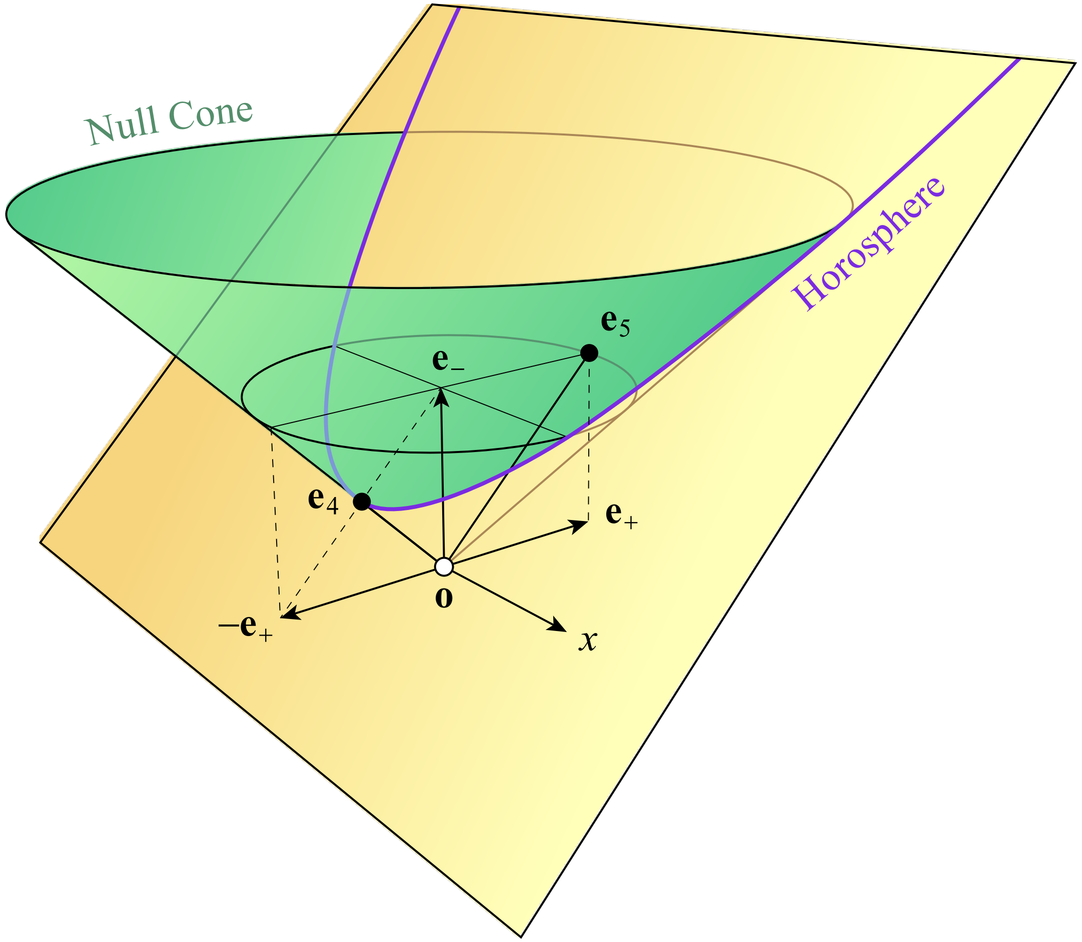
- Smaller storage requirements
  - Usually 8 floats, but can reduce to 6
- Inversion is trivial
  - Just reverse, negating six bivector components
- Better parameterization
- Better interpolation properties

Transformation Groups



Conformal Algebras

- 5D representation space for 3D geometry and motion
- Doubly projective
- Contains round objects:
  - Spheres
  - Circles
  - Dipoles
  - Round points
- Points, lines, and planes are special cases with infinite radii



Conformal Exterior Algebra

- 5D algebra modeling 3D geometry and motion

$$\mathfrak{g}_{\pm} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$

$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_- \cdot \mathbf{e}_- = -1$$

$$\mathbf{e}_+ \cdot \mathbf{e}_+ = +1$$

Conformal Exterior Algebra

- It is convenient to change the basis as follows

$$\mathbf{e}_4 = \tfrac{1}{2}(\mathbf{e}_- - \mathbf{e}_+)$$

$$\mathbf{e}_5 = \mathbf{e}_- + \mathbf{e}_+$$

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$

$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_4 \cdot \mathbf{e}_5 = -1$$

Conformal Basis Elements

Type	Grade	Basis Elements
Scalar	0	$\mathbf{1}$
Vectors	1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5$
Bivectors	2	$\mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{15}, \mathbf{e}_{25}, \mathbf{e}_{35}, \mathbf{e}_{45}$
Trivectors	3	$\mathbf{e}_{423}, \mathbf{e}_{431}, \mathbf{e}_{412}, \mathbf{e}_{321}, \mathbf{e}_{415}, \mathbf{e}_{425}, \mathbf{e}_{435}, \mathbf{e}_{235}, \mathbf{e}_{315}, \mathbf{e}_{125}$
Quadrivectors	4	$\mathbf{e}_{1234}, \mathbf{e}_{4235}, \mathbf{e}_{4315}, \mathbf{e}_{4125}, \mathbf{e}_{3215}$
Antiscalar	5	$\mathbb{1} = \mathbf{e}_{12345}$

Special Points

- $\mathbf{e}_4$ still represents the origin

- $\mathbf{e}_5$ represents the point at infinity in a stereographic projection

Flat Objects

- Everything from PGA appears in CGA with factor of $\mathbf{e}_5$

$$\mathbf{p} = p_x \mathbf{e}_{15} + p_y \mathbf{e}_{25} + p_z \mathbf{e}_{35} + p_w \mathbf{e}_{45}$$

$$\boldsymbol{l} = l_{vx} \mathbf{e}_{415} + l_{vy} \mathbf{e}_{425} + l_{vz} \mathbf{e}_{435} + l_{mx} \mathbf{e}_{235} + l_{my} \mathbf{e}_{315} + l_{mz} \mathbf{e}_{125}$$

$$\mathbf{g} = g_x \mathbf{e}_{4235} + g_y \mathbf{e}_{4315} + g_z \mathbf{e}_{4125} + g_w \mathbf{e}_{3215}$$

Round Objects

- We also have four new types of round object
  - Round points
  - Dipoles
  - Circles
  - Spheres
- Flat points, lines, and planes are special cases of dipoles, circles, and spheres that include the point at infinity

Round Point

$$\mathbf{a} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + \mathbf{e}_4 + \frac{\mathbf{p}^2 + r^2}{2} \mathbf{e}_5$$

$$\mathbf{a} = a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3 + a_w \mathbf{e}_4 + a_u \mathbf{e}_5 ,$$

Carrier Point: $a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3 + a_w \mathbf{e}_4$

Infinity: $a_u \mathbf{e}_5$

$(\text{when } a_x = a_y = a_z = a_w = 0)$

Dipole

$$\mathbf{d} = n_x\mathbf{e}_{41} + n_y\mathbf{e}_{42} + n_z\mathbf{e}_{43} + (p_y n_z - p_z n_y)\mathbf{e}_{23} + (p_z n_x - p_x n_z)\mathbf{e}_{31} + (p_x n_y - p_y n_x)\mathbf{e}_{12}$$

$$+ (\mathbf{p}\cdot\mathbf{n})(p_x\mathbf{e}_{15} + p_y\mathbf{e}_{25} + p_z\mathbf{e}_{35} + \mathbf{e}_{45}) - \frac{\mathbf{p}^2 + r^2}{2}(n_x\mathbf{e}_{15} + n_y\mathbf{e}_{25} + n_z\mathbf{e}_{35})$$

Cocarrier Normal

Cocarrier Position

$$\mathbf{d} = d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + d_{mx}\mathbf{e}_{23} + d_{my}\mathbf{e}_{31} + d_{mz}\mathbf{e}_{12} + d_{px}\mathbf{e}_{15} + d_{py}\mathbf{e}_{25} + d_{pz}\mathbf{e}_{35} + d_{pw}\mathbf{e}_{45}.$$

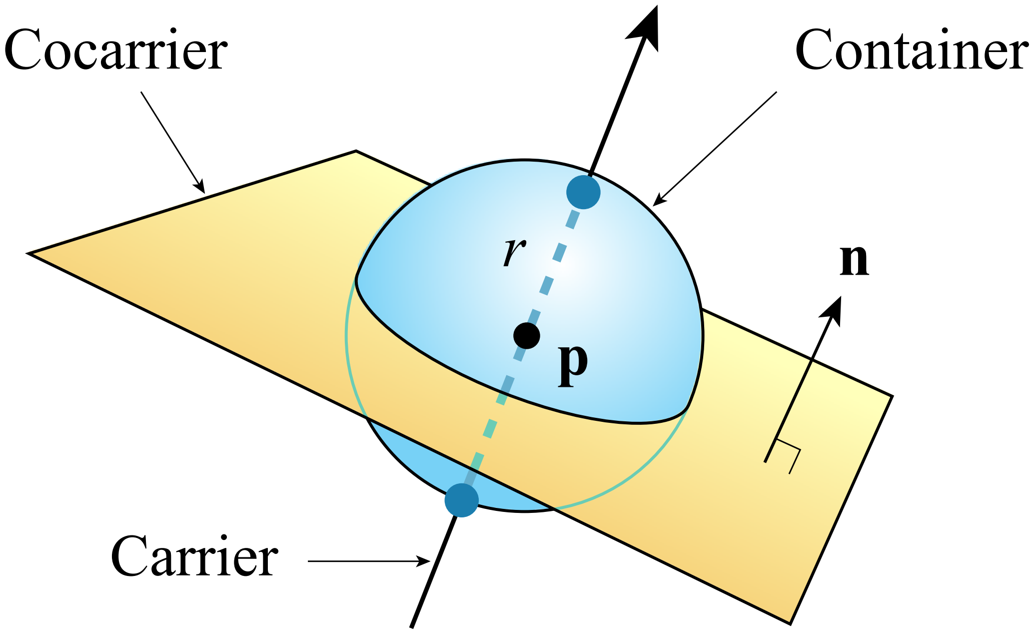
Carrier Line

Flat Point

(when $d_{vz} = d_{vy} = d_{vz} = d_{mx} = d_{my} = d_{mz} = 0$)

Dipole

- A dipole is a one-dimensional sphere



Circle

$$\mathbf{c} = n_x \mathbf{e}_{423} + n_y \mathbf{e}_{431} + n_z \mathbf{e}_{412} + (p_y n_z - p_z n_y)\mathbf{e}_{415} + (p_z n_x - p_x n_z)\mathbf{e}_{425} + (p_x n_y - p_y n_x)\mathbf{e}_{435}$$

$$+ (\mathbf{p}\cdot\mathbf{n})(p_x \mathbf{e}_{235} + p_y \mathbf{e}_{315} + p_z \mathbf{e}_{125} - \mathbf{e}_{321}) - \frac{\mathbf{p}^2 - r^2}{2}(n_x \mathbf{e}_{235} + n_y \mathbf{e}_{315} + n_z \mathbf{e}_{125})$$

Cocarrier Direction

Cocarrier Moment

$$\mathbf{c} = c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412} + c_{gw}\mathbf{e}_{321} + c_{vx}\mathbf{e}_{415} + c_{vy}\mathbf{e}_{425} + c_{vz}\mathbf{e}_{435} + c_{mx}\mathbf{e}_{235} + c_{my}\mathbf{e}_{315} + c_{mz}\mathbf{e}_{125}.$$

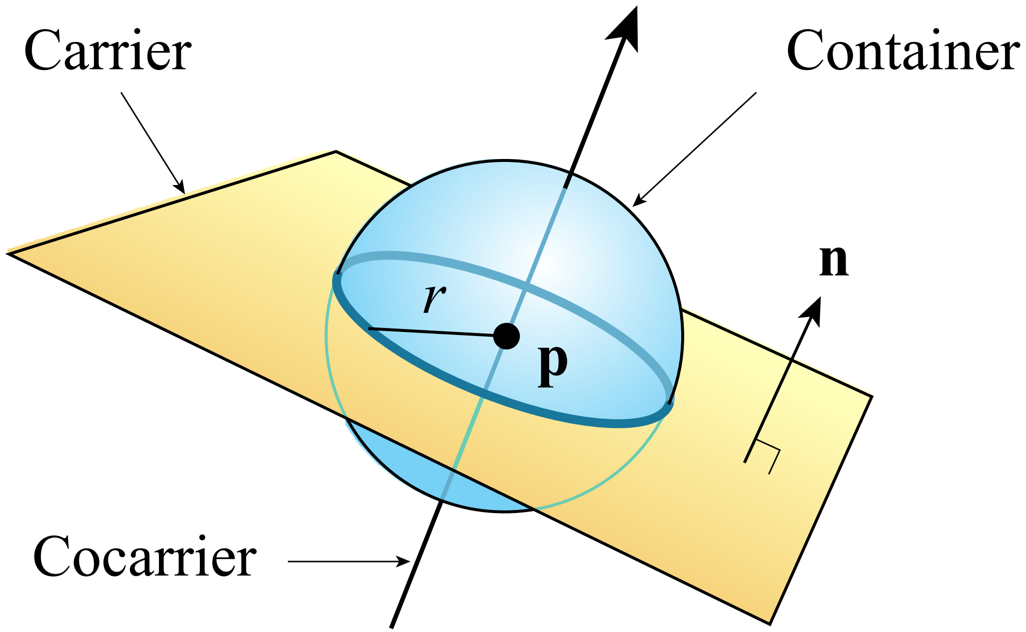
Carrier Plane

Flat Line

$(\text{when } c_{gx} = c_{gy} = c_{gz} = c_{gw} = 0)$

Circle

- A circle is a two-dimensional sphere



Sphere

$$\mathbf{s} = p_x \mathbf{e}_{4235} + p_y \mathbf{e}_{4315} + p_z \mathbf{e}_{4125} - \mathbf{e}_{1234} - \frac{\mathbf{p}^2 - r^2}{2} \mathbf{e}_{3215}$$

$$\mathbf{s} = s_u \mathbf{e}_{1234} + s_x \mathbf{e}_{4235} + s_y \mathbf{e}_{4315} + s_z \mathbf{e}_{4125} + s_w \mathbf{e}_{3215},$$

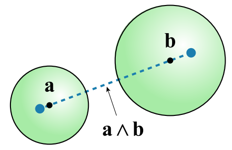
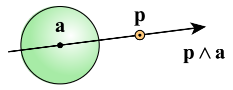
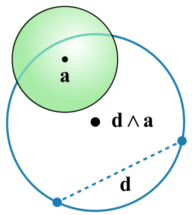
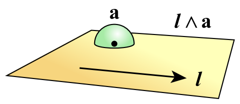
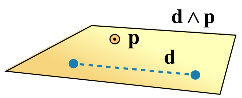
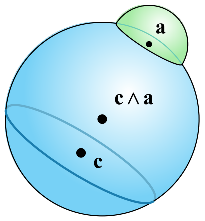
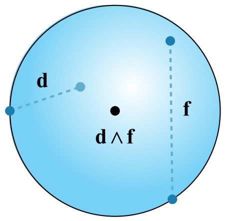
Carrier Space ($s_u \mathbf{e}_{1234}$)

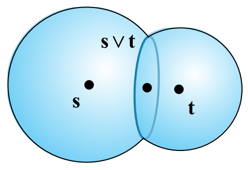
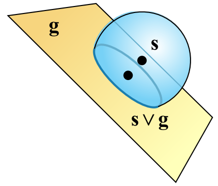
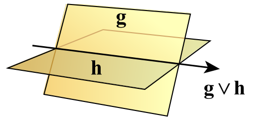
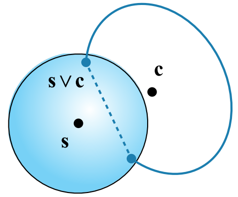
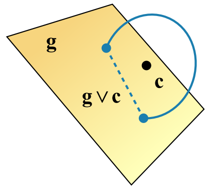
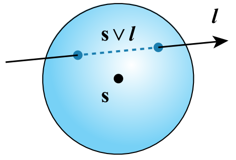
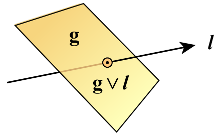
Flat Plane ($s_x \mathbf{e}_{4235} + s_y \mathbf{e}_{4315} + s_z \mathbf{e}_{4125} + s_w \mathbf{e}_{3215}$)

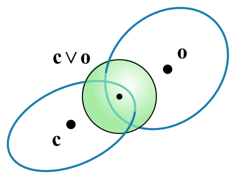
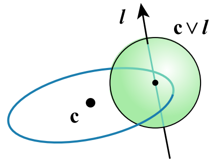
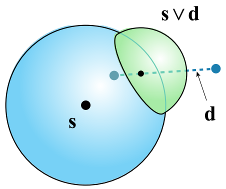
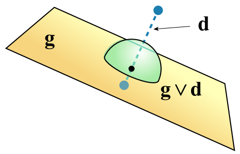
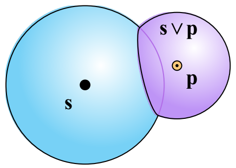
(when $s_u = 0$)

Join and Meet

- Objects joined with wedge product
- Intersection calculated with antiwedge product
- Same math as PGA

Join Operation	Illustration
Dipole containing round points **a** and **b**. $\mathbf{a} \wedge \mathbf{b} = (a_w b_x - a_x b_w)\,\mathbf{e}_{41} + (a_w b_y - a_y b_w)\,\mathbf{e}_{42} + (a_w b_z - a_z b_w)\,\mathbf{e}_{43}$ $+ (a_y b_z - a_z b_y)\,\mathbf{e}_{23} + (a_z b_x - a_x b_z)\,\mathbf{e}_{31} + (a_x b_y - a_y b_x)\,\mathbf{e}_{12}$ $+ (a_x b_u - a_u b_x)\,\mathbf{e}_{15} + (a_y b_u - a_u b_y)\,\mathbf{e}_{25}$ $+ (a_z b_u - a_u b_z)\,\mathbf{e}_{35} + (a_w b_u - a_u b_w)\,\mathbf{e}_{45}$	
Line containing flat point **p** and round point **a**. $\mathbf{p} \wedge \mathbf{a} = (p_x a_w - p_w a_x)\,\mathbf{e}_{415} + (p_z a_y - p_y a_z)\,\mathbf{e}_{235}$ $+ (p_y a_w - p_w a_y)\,\mathbf{e}_{425} + (p_x a_z - p_z a_x)\,\mathbf{e}_{315}$ $+ (p_z a_w - p_w a_z)\,\mathbf{e}_{435} + (p_y a_x - p_x a_y)\,\mathbf{e}_{125}$	
Circle containing dipole **d** and round point **a**. $\mathbf{d} \wedge \mathbf{a} = (d_{vy} a_z - d_{vz} a_y + d_{mx} a_w)\,\mathbf{e}_{423} + (d_{vz} a_x - d_{vx} a_z + d_{my} a_w)\,\mathbf{e}_{431}$ $+ (d_{vx} a_y - d_{vy} a_x + d_{mz} a_w)\,\mathbf{e}_{412} - (d_{mx} a_x + d_{my} a_y + d_{mz} a_z)\,\mathbf{e}_{321}$ $+ (d_{px} a_w - d_{pw} a_x + d_{vx} a_u)\,\mathbf{e}_{415} + (d_{pz} a_y - d_{py} a_z + d_{mx} a_u)\,\mathbf{e}_{235}$ $+ (d_{py} a_w - d_{pw} a_y + d_{vy} a_u)\,\mathbf{e}_{425} + (d_{px} a_z - d_{pz} a_x + d_{my} a_u)\,\mathbf{e}_{315}$ $+ (d_{pz} a_w - d_{pw} a_z + d_{vz} a_u)\,\mathbf{e}_{435} + (d_{py} a_x - d_{px} a_y + d_{mz} a_u)\,\mathbf{e}_{125}$	
Plane containing line ***l*** and round point **a**. $\boldsymbol{l} \wedge \mathbf{a} = (l_{vz} a_y - l_{vy} a_z - l_{mx} a_w)\,\mathbf{e}_{4235} + (l_{vx} a_z - l_{vz} a_x - l_{my} a_w)\,\mathbf{e}_{4315}$ $+ (l_{vy} a_x - l_{vx} a_y - l_{mz} a_w)\,\mathbf{e}_{4125} + (l_{mx} a_x + l_{my} a_y + l_{mz} a_z)\,\mathbf{e}_{3215}$	
Plane containing dipole **d** and flat point **p**. $\mathbf{d} \wedge \mathbf{p} = (d_{vy} p_z - d_{vz} p_y + d_{mx} p_w)\,\mathbf{e}_{4235}$ $+ (d_{vz} p_x - d_{vx} p_z + d_{my} p_w)\,\mathbf{e}_{4315}$ $+ (d_{vx} p_y - d_{vy} p_x + d_{mz} p_w)\,\mathbf{e}_{4125}$ $- (d_{mx} p_x + d_{my} p_y + d_{mz} p_z)\,\mathbf{e}_{3215}$	
Sphere containing circle **c** and round point **a**. $\mathbf{c} \wedge \mathbf{a} = -(c_{gx} a_x + c_{gy} a_y + c_{gz} a_z + c_{gw} a_w)\,\mathbf{e}_{1234}$ $+ (c_{vz} a_y - c_{vy} a_z + c_{gx} a_u - c_{mx} a_w)\,\mathbf{e}_{4235}$ $+ (c_{vx} a_z - c_{vz} a_x + c_{gy} a_u - c_{my} a_w)\,\mathbf{e}_{4315}$ $+ (c_{vy} a_x - c_{vx} a_y + c_{gz} a_u - c_{mz} a_w)\,\mathbf{e}_{4125}$ $+ (c_{mx} a_x + c_{my} a_y + c_{mz} a_z + c_{gw} a_u)\,\mathbf{e}_{3215}$	
Sphere containing dipoles **d** and **f**. $\mathbf{d} \wedge \mathbf{f} = -(d_{vx} f_{mx} + d_{vy} f_{my} + d_{vz} f_{mz} + d_{mx} f_{vx} + d_{my} f_{vy} + d_{mz} f_{vz})\,\mathbf{e}_{1234}$ $+ (d_{vy} f_{pz} - d_{vz} f_{py} + d_{pz} f_{vy} - d_{py} f_{vz} + d_{mx} f_{pw} + d_{pw} f_{mx})\,\mathbf{e}_{4235}$ $+ (d_{vz} f_{px} - d_{vx} f_{pz} + d_{px} f_{vz} - d_{pz} f_{vx} + d_{my} f_{pw} + d_{pw} f_{my})\,\mathbf{e}_{4315}$ $+ (d_{vx} f_{py} - d_{vy} f_{px} + d_{py} f_{vx} - d_{px} f_{vy} + d_{mz} f_{pw} + d_{pw} f_{mz})\,\mathbf{e}_{4125}$ $- (d_{mx} f_{px} + d_{my} f_{py} + d_{mz} f_{pz} + d_{px} f_{mx} + d_{py} f_{my} + d_{pz} f_{mz})\,\mathbf{e}_{3215}$	

Meet Operation	Illustration
Circle where spheres **s** and **t** intersect. $\mathbf{s} \vee \mathbf{t} = (s_u t_x - s_x t_u)\,\mathbf{e}_{423} + (s_u t_y - s_y t_u)\,\mathbf{e}_{431}$ $+ (s_u t_z - s_z t_u)\,\mathbf{e}_{412} + (s_u t_w - s_w t_u)\,\mathbf{e}_{321}$ $+ (s_z t_y - s_y t_z)\,\mathbf{e}_{415} + (s_x t_z - s_z t_x)\,\mathbf{e}_{425} + (s_y t_x - s_x t_y)\,\mathbf{e}_{435}$ $+ (s_x t_w - s_w t_x)\,\mathbf{e}_{235} + (s_y t_w - s_w t_y)\,\mathbf{e}_{315} + (s_z t_w - s_w t_z)\,\mathbf{e}_{125}$	
Circle where sphere **s** and plane **g** intersect. $\mathbf{s} \vee \mathbf{g} = s_u g_x\,\mathbf{e}_{423} + s_u g_y\,\mathbf{e}_{431} + s_u g_z\,\mathbf{e}_{412} + s_u g_w\,\mathbf{e}_{321}$ $+ (s_z g_y - s_y g_z)\,\mathbf{e}_{415} + (s_x g_z - s_z g_x)\,\mathbf{e}_{425} + (s_y g_x - s_x g_y)\,\mathbf{e}_{435}$ $+ (s_x g_w - s_w g_x)\,\mathbf{e}_{235} + (s_y g_w - s_w g_y)\,\mathbf{e}_{315} + (s_z g_w - s_w g_z)\,\mathbf{e}_{125}$	
Line where planes **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\,\mathbf{e}_{415} + (g_x h_w - g_w h_x)\,\mathbf{e}_{235}$ $+ (g_x h_z - g_z h_x)\,\mathbf{e}_{425} + (g_y h_w - g_w h_y)\,\mathbf{e}_{315}$ $+ (g_y h_x - g_x h_y)\,\mathbf{e}_{435} + (g_z h_w - g_w h_z)\,\mathbf{e}_{125}$	
Dipole where sphere **s** and circle **c** intersect. $\mathbf{s} \vee \mathbf{c} = (s_y c_{gz} - s_z c_{gy} + s_u c_{vx})\,\mathbf{e}_{41} + (s_w c_{gx} - s_x c_{gw} + s_u c_{mx})\,\mathbf{e}_{23}$ $+ (s_z c_{gx} - s_x c_{gz} + s_u c_{vy})\,\mathbf{e}_{42} + (s_w c_{gy} - s_y c_{gw} + s_u c_{my})\,\mathbf{e}_{31}$ $+ (s_x c_{gy} - s_y c_{gx} + s_u c_{vz})\,\mathbf{e}_{43} + (s_w c_{gz} - s_z c_{gw} + s_u c_{mz})\,\mathbf{e}_{12}$ $+ (s_z c_{my} - s_y c_{mz} + s_w c_{vx})\,\mathbf{e}_{15} + (s_x c_{mz} - s_z c_{mx} + s_w c_{vy})\,\mathbf{e}_{25}$ $+ (s_y c_{mx} - s_x c_{my} + s_w c_{vz})\,\mathbf{e}_{35} - (s_x c_{vx} + s_y c_{vy} + s_z c_{vz})\,\mathbf{e}_{45}$	
Dipole where plane **g** and circle **c** intersect. $\mathbf{g} \vee \mathbf{c} = (g_y c_{gz} - g_z c_{gy})\,\mathbf{e}_{41} + (g_w c_{gx} - g_x c_{gw})\,\mathbf{e}_{23}$ $+ (g_z c_{gx} - g_x c_{gz})\,\mathbf{e}_{42} + (g_w c_{gy} - g_y c_{gw})\,\mathbf{e}_{31}$ $+ (g_x c_{gy} - g_y c_{gx})\,\mathbf{e}_{43} + (g_w c_{gz} - g_z c_{gw})\,\mathbf{e}_{12}$ $+ (g_z c_{my} - g_y c_{mz} + g_w c_{vx})\,\mathbf{e}_{15} + (g_x c_{mz} - g_z c_{mx} + g_w c_{vy})\,\mathbf{e}_{25}$ $+ (g_y c_{mx} - g_x c_{my} + g_w c_{vz})\,\mathbf{e}_{35} - (g_x c_{vx} + g_y c_{vy} + g_z c_{vz})\,\mathbf{e}_{45}$	
Dipole where sphere **s** and line ***l*** intersect. $\mathbf{s} \vee \boldsymbol{l} = s_u l_{vx}\,\mathbf{e}_{41} + s_u l_{vy}\,\mathbf{e}_{42} + s_u l_{vz}\,\mathbf{e}_{43}$ $+ s_u l_{mx}\,\mathbf{e}_{23} + s_u l_{my}\,\mathbf{e}_{31} + s_u l_{mz}\,\mathbf{e}_{12}$ $+ (s_z l_{my} - s_y l_{mz} + s_w l_{vx})\,\mathbf{e}_{15} + (s_x l_{mz} - s_z l_{mx} + s_w l_{vy})\,\mathbf{e}_{25}$ $+ (s_y l_{mx} - s_x l_{my} + s_w l_{vz})\,\mathbf{e}_{35} - (s_x l_{vx} + s_y l_{vy} + s_z l_{vz})\,\mathbf{e}_{45}$	
Flat point where plane **g** and line ***l*** intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\,\mathbf{e}_{15} + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\,\mathbf{e}_{25}$ $+ (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\,\mathbf{e}_{35} - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\,\mathbf{e}_{45}$	

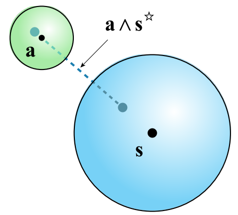
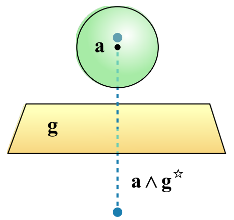
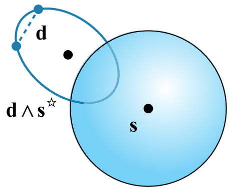
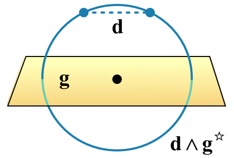
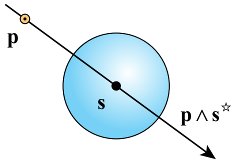
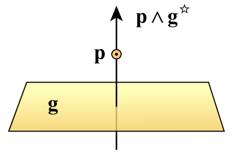
Meet Operation	Illustration
Round point contained by circles **c** and **o**. $\mathbf{c} \vee \mathbf{o} = (c_{gz} o_{my} - c_{gy} o_{mz} + c_{my} o_{gz} - c_{mz} o_{gy} + c_{vx} o_{gw} + g_{gw} o_{vx})\,\mathbf{e}_1$ $+ (c_{gx} o_{mz} - c_{gz} o_{mx} + c_{mz} o_{gx} - c_{mx} o_{gz} + c_{vy} o_{gw} + g_{gw} o_{vy})\,\mathbf{e}_2$ $+ (c_{gy} o_{mx} - c_{gx} o_{my} + c_{mx} o_{gy} - c_{my} o_{gx} + c_{vz} o_{gw} + g_{gw} o_{vz})\,\mathbf{e}_3$ $- (c_{gx} o_{vx} + c_{gy} o_{vy} + c_{gz} o_{vz} + c_{vx} o_{gx} + c_{vy} o_{gy} + c_{vz} o_{gz})\,\mathbf{e}_4$ $- (c_{mx} o_{vx} + c_{my} o_{vy} + c_{mz} o_{vz} + c_{vx} o_{mx} + c_{vy} o_{my} + c_{vz} o_{mz})\,\mathbf{e}_5$	
Round point centered on line ***l*** and contained by circle **c**. $\mathbf{c} \vee \boldsymbol{l} = (c_{gz} l_{my} - c_{gy} l_{mz} + c_{gw} l_{vx})\,\mathbf{e}_1 + (c_{gx} l_{mz} - c_{gz} l_{mx} + c_{gw} l_{vy})\,\mathbf{e}_2$ $+ (c_{gy} l_{mx} - c_{gx} l_{my} + c_{gw} l_{vz})\,\mathbf{e}_3 - (c_{gx} l_{vx} + c_{gy} l_{vy} + c_{gz} l_{vz})\,\mathbf{e}_4$ $- (c_{mx} l_{vx} + c_{my} l_{vy} + c_{mz} l_{vz} + c_{vx} l_{mx} + c_{vy} l_{my} + c_{vz} l_{mz})\,\mathbf{e}_5$	
Round point contained by sphere **s** and dipole **d**. $\mathbf{s} \vee \mathbf{d} = (s_y d_{mz} - s_z d_{my} - s_w d_{vx} + s_u d_{px})\,\mathbf{e}_1$ $+ (s_z d_{mx} - s_x d_{mz} - s_w d_{vy} + s_u d_{py})\,\mathbf{e}_2$ $+ (s_x d_{my} - s_y d_{mx} - s_w d_{vz} + s_u d_{pz})\,\mathbf{e}_3$ $+ (s_x d_{vx} + s_y d_{vy} + s_z d_{vz} + s_u d_{pw})\,\mathbf{e}_4$ $- (s_x d_{px} + s_y d_{py} + s_z d_{pz} + s_w d_{pw})\,\mathbf{e}_5$	
Round point centered in plane **g** and contained by dipole **d**. $\mathbf{g} \vee \mathbf{d} = (g_y d_{mz} - g_z d_{my} - g_w d_{vx})\,\mathbf{e}_1$ $+ (g_z d_{mx} - g_x d_{mz} - g_w d_{vy})\,\mathbf{e}_2$ $+ (g_x d_{my} - g_y d_{mx} - g_w d_{vz})\,\mathbf{e}_3$ $+ (g_x d_{vx} + g_y d_{vy} + d_z d_{vz})\,\mathbf{e}_4$ $- (g_x d_{px} + g_y d_{py} + g_z d_{pz} + g_w d_{pw})\,\mathbf{e}_5$	
Round point centered at flat point **p** and contained by sphere **s**. $\mathbf{s} \vee \mathbf{p} = s_u p_x\,\mathbf{e}_1 + s_u p_y\,\mathbf{e}_2 + s_u p_z\,\mathbf{e}_3 + s_u p_w\,\mathbf{e}_4$ $- (s_x p_x + s_y p_y + s_z p_z + s_w p_w)\,\mathbf{e}_5$	

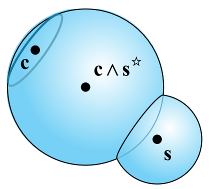
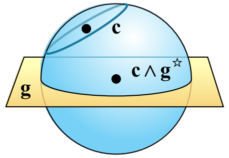
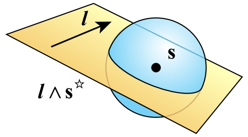
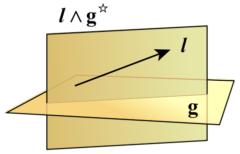
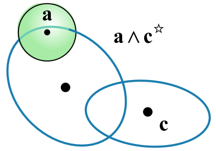
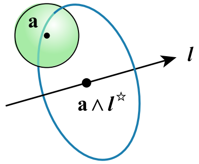
Weight Expansion

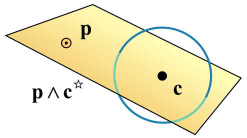
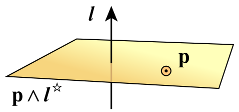
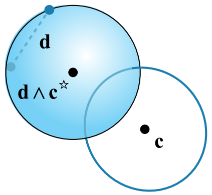
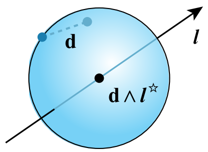
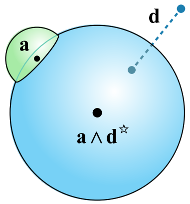
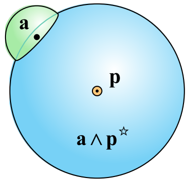
- Weight expansion calculates geometry containing one object which is orthogonal to another object

- Same math as PGA

- Projections travel along spheres, however
  - There are no ellipsoidal shapes in CGA

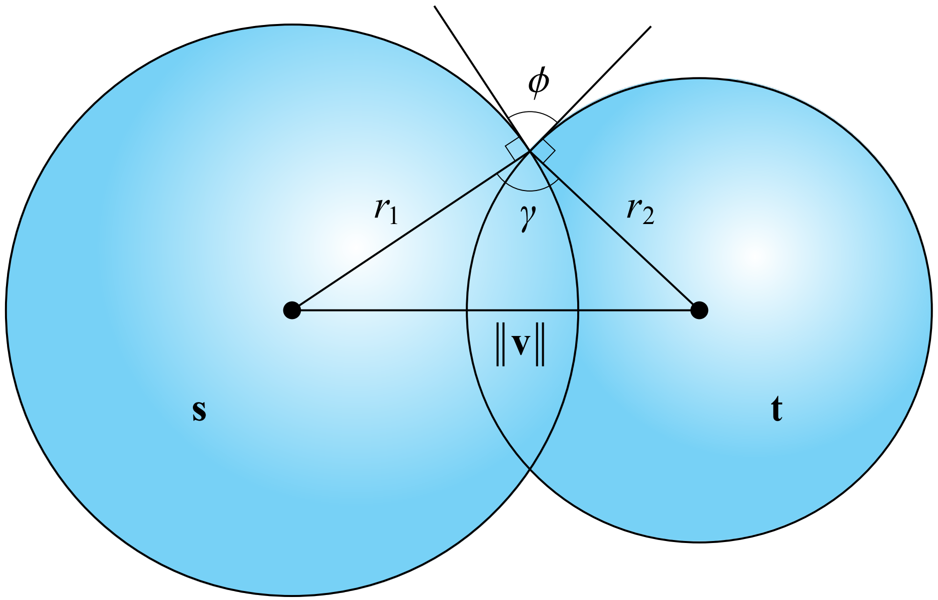
Expansion Operation	Illustration
Dipole containing round point **a** and orthogonal to sphere **s**. $\mathbf{a}\wedge\mathbf{s}^{\star}=(a_x s_u+a_w s_x)\,\mathbf{e}_{41}+(a_y s_z-a_z s_y)\,\mathbf{e}_{23}$ $+(a_y s_u+a_w s_y)\,\mathbf{e}_{42}+(a_z s_x-a_x s_z)\,\mathbf{e}_{31}$ $+(a_z s_u+a_w s_z)\,\mathbf{e}_{43}+(a_x s_y-a_y s_x)\,\mathbf{e}_{12}$ $-(a_x s_w+a_u s_x)\,\mathbf{e}_{15}-(a_y s_w+a_u s_y)\,\mathbf{e}_{25}$ $-(a_z s_w+a_u s_z)\,\mathbf{e}_{35}+(a_u s_u-a_w s_w)\,\mathbf{e}_{45}$	
Dipole containing round point **a** and orthogonal to plane **g**. $\mathbf{a}\wedge\mathbf{g}^{\star}=a_w g_x\,\mathbf{e}_{41}+(a_y g_z-a_z g_y)\,\mathbf{e}_{23}$ $+a_w g_y\,\mathbf{e}_{42}+(a_z g_x-a_x g_z)\,\mathbf{e}_{31}$ $+a_w g_z\,\mathbf{e}_{43}+(a_x g_y-a_y g_x)\,\mathbf{e}_{12}$ $-(a_x g_w+a_u g_x)\,\mathbf{e}_{15}-(a_y g_w+a_u g_y)\,\mathbf{e}_{25}$ $-(a_z g_w+a_u g_z)\,\mathbf{e}_{35}-a_w g_w\,\mathbf{e}_{45}$	
Circle containing dipole **d** and orthogonal to sphere **s**. $\mathbf{d}\wedge\mathbf{s}^{\star}=(d_{vy}s_z-d_{vz}s_y-d_{mx}s_u)\,\mathbf{e}_{423}+(d_{vz}s_x-d_{vx}s_z-d_{my}s_u)\,\mathbf{e}_{431}$ $+(d_{vx}s_y-d_{vy}s_x-d_{mz}s_u)\,\mathbf{e}_{412}-(d_{mx}s_x+d_{my}s_y+d_{mz}s_z)\,\mathbf{e}_{321}$ $-(d_{vx}s_w+d_{pw}s_x+d_{px}s_u)\,\mathbf{e}_{415}+(d_{pz}s_y-d_{py}s_z-d_{mx}s_w)\,\mathbf{e}_{235}$ $-(d_{vy}s_w+d_{pw}s_y+d_{py}s_u)\,\mathbf{e}_{425}+(d_{px}s_z-d_{pz}s_x-d_{my}s_w)\,\mathbf{e}_{315}$ $-(d_{vz}s_w+d_{pw}s_z+d_{pz}s_u)\,\mathbf{e}_{435}+(d_{py}s_x-d_{px}s_y-d_{mz}s_w)\,\mathbf{e}_{125}$	
Circle containing dipole **d** and orthogonal to plane **g**. $\mathbf{d}\wedge\mathbf{g}^{\star}=(d_{vy}g_z-d_{vz}g_y)\,\mathbf{e}_{423}+(d_{vz}g_x-d_{vx}g_z)\,\mathbf{e}_{431}$ $+(d_{vx}g_y-d_{vy}g_x)\,\mathbf{e}_{412}-(d_{mx}g_x+d_{my}g_y+d_{mz}g_z)\,\mathbf{e}_{321}$ $-(d_{vx}g_w+d_{pw}g_x)\,\mathbf{e}_{415}+(d_{pz}g_y-d_{py}g_z-d_{mx}g_w)\,\mathbf{e}_{235}$ $-(d_{vy}g_w+d_{pw}g_y)\,\mathbf{e}_{425}+(d_{px}g_z-d_{pz}g_x-d_{my}g_w)\,\mathbf{e}_{315}$ $-(d_{vz}g_w+d_{pw}g_z)\,\mathbf{e}_{435}+(d_{py}g_x-d_{px}g_y-d_{mz}g_w)\,\mathbf{e}_{125}$	
Line containing flat point **p** and orthogonal to sphere **s**. $\mathbf{p}\wedge\mathbf{s}^{\star}=-(p_w s_x+p_x s_u)\,\mathbf{e}_{415}+(p_z s_y-p_y s_z)\,\mathbf{e}_{235}$ $-(p_w s_y+p_y s_u)\,\mathbf{e}_{425}+(p_x s_z-p_z s_x)\,\mathbf{e}_{315}$ $-(p_w s_z+p_z s_u)\,\mathbf{e}_{435}+(p_y s_x-p_x s_y)\,\mathbf{e}_{125}$	
Line containing flat point **p** and orthogonal to plane **g**. $\mathbf{p}\wedge\mathbf{g}^{\star}=-p_w g_x\,\mathbf{e}_{415}+(p_z g_y-p_y g_z)\,\mathbf{e}_{235}$ $-p_w g_y\,\mathbf{e}_{425}+(p_x g_z-p_z g_x)\,\mathbf{e}_{315}$ $-p_w g_z\,\mathbf{e}_{435}+(p_y g_x-p_x g_y)\,\mathbf{e}_{125}$	

Expansion Operation	Illustration
Sphere containing circle **c** and orthogonal to sphere **s**. $\mathbf{c}\wedge\mathbf{s}^{\star}=(c_{gw}s_u-c_{gx}s_x-c_{gy}s_y-c_{gz}s_z)\,\mathbf{e}_{1234}$ $+(c_{vz}s_y-c_{vy}s_z+c_{mx}s_u-c_{gx}s_w)\,\mathbf{e}_{4235}$ $+(c_{vx}s_z-c_{vz}s_x+c_{my}s_u-c_{gy}s_w)\,\mathbf{e}_{4315}$ $+(c_{vy}s_x-c_{vx}s_y+c_{mz}s_u-c_{gz}s_w)\,\mathbf{e}_{4125}$ $+(c_{mx}s_x+c_{my}s_y+c_{mz}s_z-c_{gw}s_w)\,\mathbf{e}_{3215}$	
Sphere containing circle **c** and orthogonal to plane **g**. $\mathbf{c}\wedge\mathbf{g}^{\star}=-(c_{gx}g_x+c_{gy}g_y+c_{gz}g_z)\,\mathbf{e}_{1234}$ $+(c_{vz}g_y-c_{vy}g_z-c_{gx}g_w)\,\mathbf{e}_{4235}$ $+(c_{vx}g_z-c_{vz}g_x-c_{gy}g_w)\,\mathbf{e}_{4315}$ $+(c_{vy}g_x-c_{vx}g_y-c_{gz}g_w)\,\mathbf{e}_{4125}$ $+(c_{mx}g_x+c_{my}g_y+c_{mz}g_z-c_{gw}g_w)\,\mathbf{e}_{3215}$	
Plane containing line ***l*** and orthogonal to sphere **s**. $\boldsymbol{l}\wedge\mathbf{s}^{\star}=(l_{vz}s_y-l_{vy}s_z+l_{mx}s_u)\,\mathbf{e}_{4235}+(l_{vx}s_z-l_{vz}s_x+l_{my}s_u)\,\mathbf{e}_{4315}$ $+(l_{vy}s_x-l_{vx}s_y+l_{mz}s_u)\,\mathbf{e}_{4125}+(l_{mx}s_x+l_{my}s_y+l_{mz}s_z)\,\mathbf{e}_{3215}$	
Plane containing line ***l*** and orthogonal to plane **g**. $\boldsymbol{l}\wedge\mathbf{g}^{\star}=(l_{vz}g_y-l_{vy}g_z)\,\mathbf{e}_{4235}+(l_{vx}g_z-l_{vz}g_x)\,\mathbf{e}_{4315}$ $+(l_{vy}g_x-l_{vx}g_y)\,\mathbf{e}_{4125}+(l_{mx}g_x+l_{my}g_y+l_{mz}g_z)\,\mathbf{e}_{3215}$	
Circle containing round point **a** and orthogonal to circle **c**. $\mathbf{a}\wedge\mathbf{c}^{\star}=(a_y c_{gz}-a_z c_{gy}-a_w c_{vx})\,\mathbf{e}_{423}+(a_z c_{gx}-a_x c_{gz}-a_w c_{vy})\,\mathbf{e}_{431}$ $+(a_x c_{gy}-a_y c_{gx}-a_w c_{vz})\,\mathbf{e}_{412}+(a_x c_{vx}+a_y c_{vy}+a_z c_{vz})\,\mathbf{e}_{321}$ $-(a_x c_{gw}+a_w c_{mx}+a_u c_{gx})\,\mathbf{e}_{415}+(a_z c_{my}-a_y c_{mz}-a_u c_{vx})\,\mathbf{e}_{235}$ $-(a_y c_{gw}+a_w c_{my}+a_u c_{gy})\,\mathbf{e}_{425}+(a_x c_{mz}-a_z c_{mx}-a_u c_{vy})\,\mathbf{e}_{315}$ $-(a_z c_{gw}+a_w c_{mz}+a_u c_{gz})\,\mathbf{e}_{435}+(a_y c_{mx}-a_x c_{my}-a_u c_{vz})\,\mathbf{e}_{125}$	
Circle containing round point **a** and orthogonal to line ***l***. $\mathbf{a}\wedge\boldsymbol{l}^{\star}=-a_w l_{vx}\,\mathbf{e}_{423}-a_w l_{vy}\,\mathbf{e}_{431}-a_w l_{vz}\,\mathbf{e}_{412}$ $+(a_x l_{vx}+a_y l_{vy}+a_z l_{vz})\,\mathbf{e}_{321}$ $-a_w l_{mx}\,\mathbf{e}_{415}+(a_z l_{my}-a_y l_{mz}-a_u l_{vx})\,\mathbf{e}_{235}$ $-a_w l_{my}\,\mathbf{e}_{425}+(a_x l_{mz}-a_z l_{mx}-a_u l_{vy})\,\mathbf{e}_{315}$ $-a_w l_{mz}\,\mathbf{e}_{435}+(a_y l_{mx}-a_x l_{my}-a_u l_{vz})\,\mathbf{e}_{125}$	

Expansion Operation	Illustration
Plane containing flat point **p** and orthogonal to circle **c**. $\mathbf{p} \wedge \mathbf{c}^{\star} = (p_y c_{gz} - p_z c_{gy} - p_w c_{vx})\,\mathbf{e}_{4235}$ $+ (p_z c_{gx} - p_x c_{gz} - p_w c_{vy})\,\mathbf{e}_{4315}$ $+ (p_x c_{gy} - p_y c_{gx} - p_w c_{vz})\,\mathbf{e}_{4125}$ $+ (p_x c_{vx} + p_y c_{vy} + p_z c_{vz})\,\mathbf{e}_{3215}$	
Plane containing flat point **p** and orthogonal to line ***l***. $\mathbf{p} \wedge \boldsymbol{l}^{\star} = -p_w l_{vx}\,\mathbf{e}_{4235} - p_w l_{vy}\,\mathbf{e}_{4315} - p_w l_{vz}\,\mathbf{e}_{4125}$ $+ (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\,\mathbf{e}_{3215}$	
Sphere containing dipole **d** and orthogonal to circle **c**. $\mathbf{d} \wedge \mathbf{c}^{\star} = (d_{vx} c_{vx} + d_{vy} c_{vy} + d_{vz} c_{vz} + d_{mx} c_{gx} + d_{my} c_{gy} + d_{mz} c_{gz})\,\mathbf{e}_{1234}$ $+ (d_{vz} c_{my} - d_{vy} c_{mz} - d_{pw} c_{vx} + d_{py} c_{gz} - d_{pz} c_{gy} + d_{mx} c_{gw})\,\mathbf{e}_{4235}$ $+ (d_{vx} c_{mz} - d_{vz} c_{mx} - d_{pw} c_{vy} + d_{pz} c_{gx} - d_{px} c_{gz} + d_{my} c_{gw})\,\mathbf{e}_{4315}$ $+ (d_{vy} c_{mx} - d_{vx} c_{my} - d_{pw} c_{vz} + d_{px} c_{gy} - d_{py} c_{gx} + d_{mz} c_{gw})\,\mathbf{e}_{4125}$ $+ (d_{px} c_{vx} + d_{py} c_{vy} + d_{pz} c_{vz} + d_{mx} c_{mx} + d_{my} c_{my} + d_{mz} c_{mz})\,\mathbf{e}_{3215}$	
Sphere containing dipole **d** and orthogonal to line ***l***. $\mathbf{d} \wedge \boldsymbol{l}^{\star} = (d_{vx} l_{vx} + d_{vy} l_{vy} + d_{vz} l_{vz})\,\mathbf{e}_{1234}$ $+ (d_{vz} l_{my} - d_{vy} l_{mz} - d_{pw} l_{vx})\,\mathbf{e}_{4235}$ $+ (d_{vx} l_{mz} - d_{vz} l_{mx} - d_{pw} l_{vy})\,\mathbf{e}_{4315}$ $+ (d_{vy} l_{mx} - d_{vx} l_{my} - d_{pw} l_{vz})\,\mathbf{e}_{4125}$ $+ (d_{px} l_{vx} + d_{py} l_{vy} + d_{pz} l_{vz} + d_{mx} l_{mx} + d_{my} l_{my} + d_{mz} l_{mz})\,\mathbf{e}_{3215}$	
Sphere containing round point **a** and orthogonal to dipole **d**. $\mathbf{a} \wedge \mathbf{d}^{\star} = (a_x d_{vx} + a_y d_{vy} + a_z d_{vz} - a_w d_{pw})\,\mathbf{e}_{1234}$ $+ (a_z d_{my} - a_y d_{mz} + a_w d_{px} - a_u d_{vx})\,\mathbf{e}_{4235}$ $+ (a_x d_{mz} - a_z d_{mx} + a_w d_{py} - a_u d_{vy})\,\mathbf{e}_{4315}$ $+ (a_y d_{mx} - a_x d_{my} + a_w d_{pz} - a_u d_{vz})\,\mathbf{e}_{4125}$ $+ (a_u d_{pw} - a_x d_{px} - a_y d_{py} - a_z d_{pz})\,\mathbf{e}_{3215}$	
Sphere containing round point **a** and centered at flat point **p**. $\mathbf{a} \wedge \mathbf{p}^{\star} = -a_w p_w\,\mathbf{e}_{1234} + a_w p_x\,\mathbf{e}_{4235} + a_w p_y\,\mathbf{e}_{4315} + a_w p_z\,\mathbf{e}_{4125}$ $+ (a_u p_w - a_x p_x - a_y p_y - a_z p_z)\,\mathbf{e}_{3215}$	

Dot Products

- Dot product between two spheres is product of radii multiplied by cosine of angle between tangent planes where they intersect



Law of cosines

$$\mathbf{v}^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\gamma$$

$$\mathbf{s} \bullet \mathbf{t} = \tfrac{1}{2}\left(\mathbf{v}^2 - r_1^2 - r_2^2\right) = -r_1 r_2 \cos\gamma = r_1 r_2 \cos\phi$$

Conformal Motions

- All motions of PGA transfer to CGA with factor of $\mathbf{e}_5$

- General screw motion:

$$\mathbf{Q} = Q_{vx}\,\mathbf{e}_{415} + Q_{vy}\,\mathbf{e}_{425} + Q_{vz}\,\mathbf{e}_{435} + Q_{vw}\,\mathbb{1} + Q_{mx}\,\mathbf{e}_{235} + Q_{my}\,\mathbf{e}_{315} + Q_{mz}\,\mathbf{e}_{125} + Q_{mw}\,\mathbf{e}_{5}$$

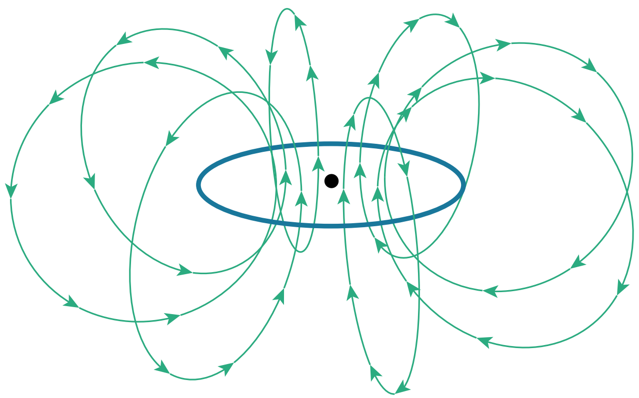
- General rotoreflection:

$$\mathbf{F} = F_{px}\,\mathbf{e}_{15} + F_{py}\,\mathbf{e}_{25} + F_{pz}\,\mathbf{e}_{35} + F_{pw}\,\mathbf{e}_{45} + F_{gx}\,\mathbf{e}_{4235} + F_{gy}\,\mathbf{e}_{4315} + F_{gz}\,\mathbf{e}_{4125} + F_{gw}\,\mathbf{e}_{3215}$$

Conformal Motions

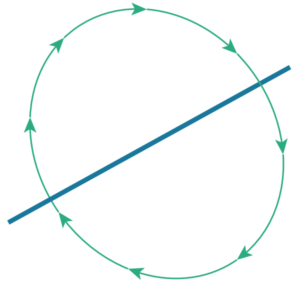
Real Circle / Elliptic Rotation

$\mathbf{R} = \mathbf{c}\sin\phi + \mathbb{1}\cos\phi$



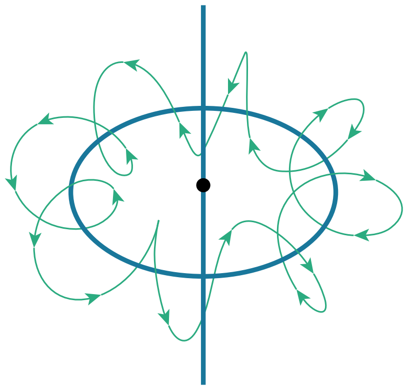
Flat Line / Rotation

$\mathbf{R} = \boldsymbol{l}\sin\phi + \mathbb{1}\cos\phi$



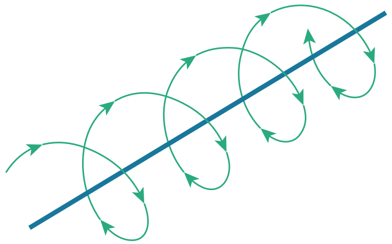
Real Circle + Line

Twisted Elliptic Rotation



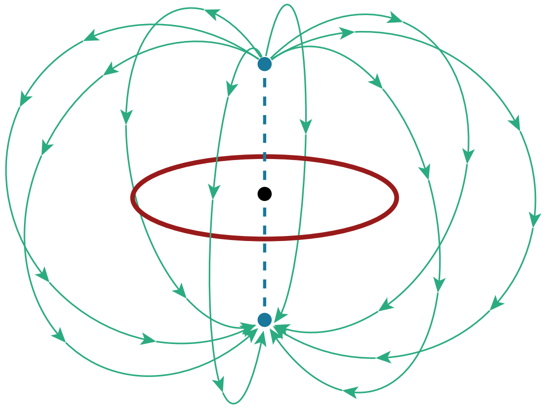
Line or Point in Horizon + Line

Twisted Rotation / Screw Motion



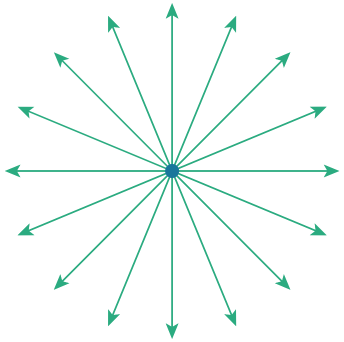
Imaginary Circle / Hyperbolic Rotation

$\mathbf{R} = \mathbf{c}\sinh\phi + \mathbb{1}\cosh\phi$



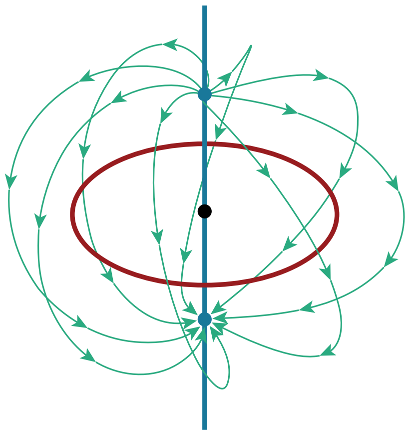
Dual Flat Point / Dilation

$\mathbf{D} = \frac{1-\sigma}{1+\sigma}\mathbf{p}^{\star} + \mathbb{1}$



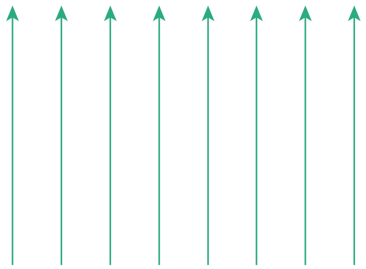
Imaginary Circle + Line

Twisted Hyperbolic Rotation



Line or Point in Horizon /Translation

$\mathbf{T} = \mathbf{v}^{\star} + \mathbb{1}$



Conformal Motions

- Operators are equivalent to 5 × 5 matrices
- Simple translation example:

$$\mathbf{T} = \tau_x \mathbf{e}_{235} + \tau_y \mathbf{e}_{315} + \tau_z \mathbf{e}_{125} + \mathbb{1}$$

$$\mathbf{t} = 2\boldsymbol{\tau}$$

$$\begin{bmatrix} 1 & 0 & 0 & t_x & 0 \\ 0 & 1 & 0 & t_y & 0 \\ 0 & 0 & 1 & t_z & 0 \\ 0 & 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & \frac{1}{2}\mathbf{t}^2 & 1 \end{bmatrix}$$

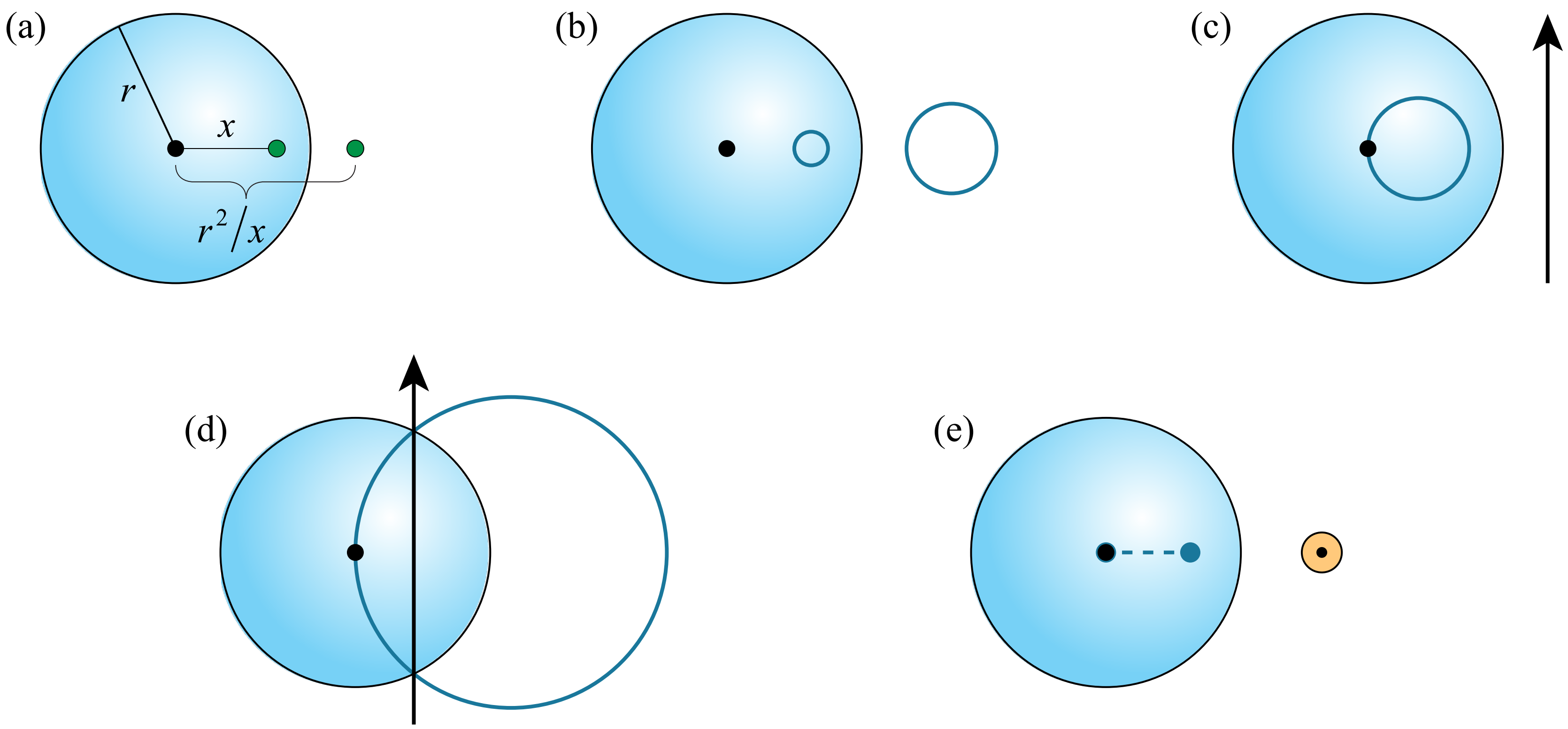
Translation

- Computation gets somewhat absurd
- Would be easier to store object as center, radius, attitude
- Rebuild CGA form as needed

Type	Translation Formula
Flat point **p**	$\mathbf{T} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{T}} = (p_x + 2\tau_x p_w)\mathbf{e}_{15} + (p_y + 2\tau_y p_w)\mathbf{e}_{25} + (p_z + 2\tau_z p_w)\mathbf{e}_{35} + p_w \mathbf{e}_{45}$
Line ***l***	$\mathbf{T} \vee \boldsymbol{l} \vee \underset{\sim}{\mathbf{T}} = l_{vx}\mathbf{e}_{415} + l_{vy}\mathbf{e}_{425} + l_{vz}\mathbf{e}_{435} + [l_{mx} + 2(\tau_y l_{vz} - \tau_z l_{vy})]\mathbf{e}_{235}$ $+ [l_{my} + 2(\tau_z l_{vx} - \tau_x l_{vz})]\mathbf{e}_{315} + [l_{mz} + 2(\tau_x l_{vy} - \tau_y l_{vx})]\mathbf{e}_{125}$
Plane **g**	$\mathbf{T} \vee \mathbf{g} \vee \underset{\sim}{\mathbf{T}} = g_x \mathbf{e}_{4235} + g_y \mathbf{e}_{4315} + g_z \mathbf{e}_{4125} + (g_w - 2\boldsymbol{\tau} \cdot \mathbf{g}_{xyz})\mathbf{e}_{3215}$
Round point **a**	$\mathbf{T} \vee \mathbf{a} \vee \underset{\sim}{\mathbf{T}} = (a_x + 2\tau_x a_w)\mathbf{e}_1 + (a_y + 2\tau_y a_w)\mathbf{e}_2 + (a_z + 2\tau_z a_w)\mathbf{e}_3 + a_w \mathbf{e}_4$ $+ \left(a_u + 2\boldsymbol{\tau} \cdot \mathbf{a}_{xyz} + 2\boldsymbol{\tau}^2 a_w\right)\mathbf{e}_5$
Dipole **d**	$\mathbf{T} \vee \mathbf{d} \vee \underset{\sim}{\mathbf{T}} = d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + [d_{mx} + 2(\tau_y d_{vz} - \tau_z d_{vy})]\mathbf{e}_{23}$ $+ [d_{my} + 2(\tau_z d_{vx} - \tau_x d_{vz})]\mathbf{e}_{31} + [d_{mz} + 2(\tau_x d_{vy} - \tau_y d_{vx})]\mathbf{e}_{12}$ $+ [d_{px} + 2(\tau_y d_{mz} - \tau_z d_{my} + \tau_x d_{pw} + 2\tau_x \boldsymbol{\tau} \cdot \mathbf{d}_{\mathbf{v}} - \boldsymbol{\tau}^2 d_{vx})]\mathbf{e}_{15}$ $+ [d_{py} + 2(\tau_z d_{mx} - \tau_x d_{mz} + \tau_y d_{pw} + 2\tau_y \boldsymbol{\tau} \cdot \mathbf{d}_{\mathbf{v}} - \boldsymbol{\tau}^2 d_{vy})]\mathbf{e}_{25}$ $+ [d_{pz} + 2(\tau_x d_{my} - \tau_y d_{mx} + \tau_z d_{pw} + 2\tau_z \boldsymbol{\tau} \cdot \mathbf{d}_{\mathbf{v}} - \boldsymbol{\tau}^2 d_{vz})]\mathbf{e}_{35}$ $+ (d_{pw} + 2\boldsymbol{\tau} \cdot \mathbf{d}_{\mathbf{v}})\mathbf{e}_{45}$
Circle **c**	$\mathbf{T} \vee \mathbf{c} \vee \underset{\sim}{\mathbf{T}} = c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412}$ $+ (c_{gw} - 2\boldsymbol{\tau} \cdot \mathbf{c}_{gxyz})\mathbf{e}_{321} + [c_{vx} + 2(\tau_y c_{gz} - \tau_z c_{gy})]\mathbf{e}_{415}$ $+ [c_{vy} + 2(\tau_z c_{gx} - \tau_x c_{gz})]\mathbf{e}_{425} + [c_{vz} + 2(\tau_x c_{gy} - \tau_y c_{gx})]\mathbf{e}_{435}$ $+ [c_{mx} + 2(\tau_y c_{vz} - \tau_z c_{vy} - \tau_x c_{gw} + 2\tau_x \boldsymbol{\tau} \cdot \mathbf{c}_{gxyz} - \boldsymbol{\tau}^2 c_{gx})]\mathbf{e}_{235}$ $+ [c_{my} + 2(\tau_z c_{vx} - \tau_x c_{vz} - \tau_y c_{gw} + 2\tau_y \boldsymbol{\tau} \cdot \mathbf{c}_{gxyz} - \boldsymbol{\tau}^2 c_{gy})]\mathbf{e}_{315}$ $+ [c_{mz} + 2(\tau_x c_{vy} - \tau_y c_{vx} - \tau_z c_{gw} + 2\tau_z \boldsymbol{\tau} \cdot \mathbf{c}_{gxyz} - \boldsymbol{\tau}^2 c_{gz})]\mathbf{e}_{125}$
Sphere **s**	$\mathbf{T} \vee \mathbf{s} \vee \underset{\sim}{\mathbf{T}} = s_u \mathbf{e}_{1234} + (s_x - 2\tau_x s_u)\mathbf{e}_{4235} + (s_y - 2\tau_y s_u)\mathbf{e}_{4315} + (s_z - 2\tau_z s_u)\mathbf{e}_{4125}$ $+ \left(s_w - 2\boldsymbol{\tau} \cdot \mathbf{s}_{xyz} + 2\boldsymbol{\tau}^2 s_u\right)\mathbf{e}_{3215}$

Sphere Inversion

- In CGA, reflections across planes generalize to reflections through spheres



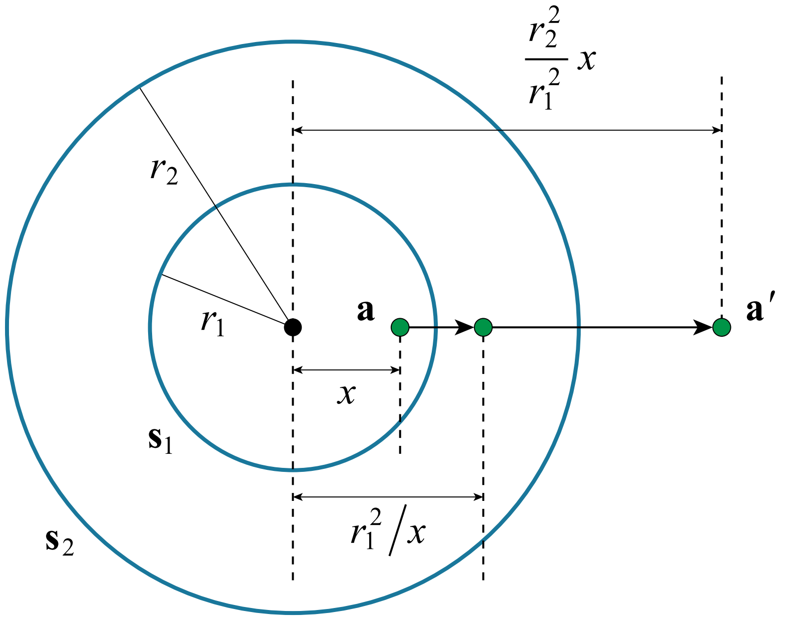
Sphere Inversion

- For sphere of radius *r* centered at $\left(m_x, m_y, m_z\right)$, points are transformed by

$$\begin{bmatrix} r^2 - 2m_x^2 & -2m_x m_y & -2m_x m_z & \left(\mathbf{m}^2 - r^2\right) m_x & 2m_x \\ -2m_x m_y & r^2 - 2m_y^2 & -2m_y m_z & \left(\mathbf{m}^2 - r^2\right) m_y & 2m_y \\ -2m_x m_z & -2m_y m_z & r^2 - 2m_z^2 & \left(\mathbf{m}^2 - r^2\right) m_z & 2m_z \\ -2m_x & -2m_y & -2m_z & \mathbf{m}^2 & 2 \\ -\left(\mathbf{m}^2 - r^2\right) m_x & -\left(\mathbf{m}^2 - r^2\right) m_y & -\left(\mathbf{m}^2 - r^2\right) m_z & \frac{1}{2}\left(\mathbf{m}^2 - r^2\right)^2 & \mathbf{m}^2 \end{bmatrix}$$

Dilation

- Translation results from reflections across two parallel planes
- This generalizes to reflections through two concentric spheres
- Result is a dilation about the center of the spheres



Dilation

- Operator that dilates by factor σ about center (m_x, m_y, m_z)

$$\mathbf{D} = \frac{1-\sigma}{2}\left(m_x \mathbf{e}_{235} + m_y \mathbf{e}_{315} + m_z \mathbf{e}_{125} - \mathbf{e}_{321}\right) + \frac{1+\sigma}{2}\mathbb{1}$$

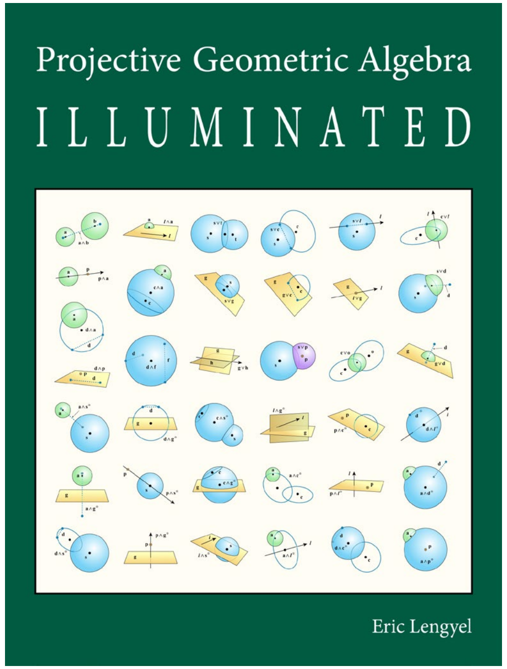
$$\begin{bmatrix} \sigma & 0 & 0 & (1-\sigma)\,m_x & 0 \\ 0 & \sigma & 0 & (1-\sigma)\,m_y & 0 \\ 0 & 0 & \sigma & (1-\sigma)\,m_z & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \sigma(1-\sigma)\,m_x & \sigma(1-\sigma)\,m_y & \sigma(1-\sigma)\,m_z & \frac{1}{2}(1-\sigma)^2\mathbf{m}^2 & \sigma^2 \end{bmatrix}$$

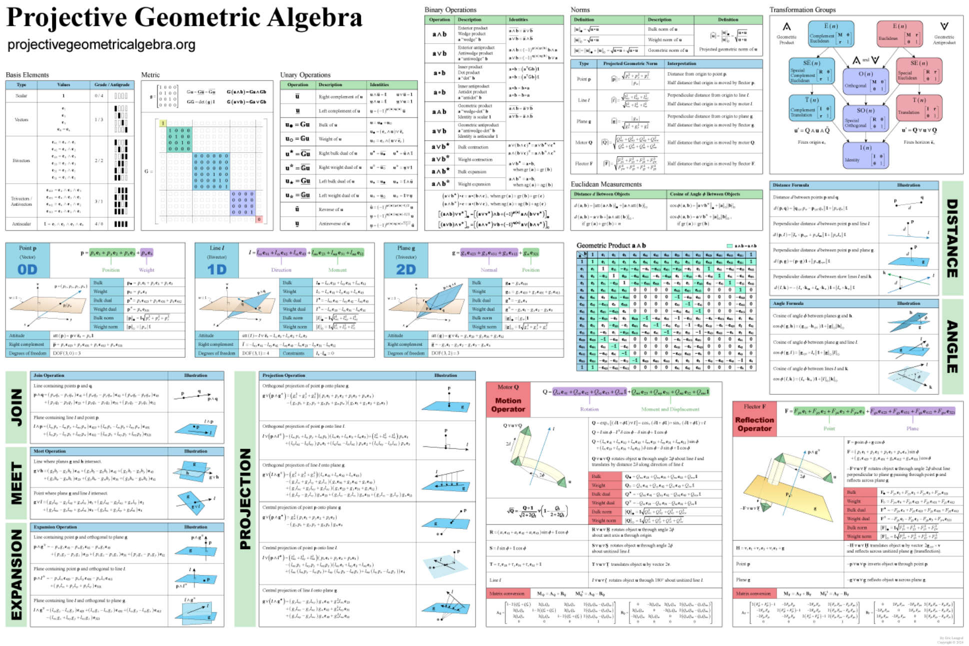
Dilation

Type	Dilation Formula
Flat point $\mathbf{p}$	$\mathbf{D} \mathbin{\dot\vee} \mathbf{p} \mathbin{\dot\vee} \underset{\sim}{\mathbf{D}} = \left[\sigma^2 p_x + \sigma(1-\sigma) m_x p_w\right]\mathbf{e}_{15} + \left[\sigma^2 p_y + \sigma(1-\sigma) m_y p_w\right]\mathbf{e}_{25}$ $+\left[\sigma^2 p_z + \sigma(1-\sigma) m_z p_w\right]\mathbf{e}_{35} + \sigma p_w \mathbf{e}_{45}$
Line $\boldsymbol{l}$	$\mathbf{D} \mathbin{\dot\vee} \boldsymbol{l} \mathbin{\dot\vee} \underset{\sim}{\mathbf{D}} = \sigma l_{vx}\mathbf{e}_{415} + \sigma l_{vy}\mathbf{e}_{425} + \sigma l_{vz}\mathbf{e}_{435}$ $+\left[\sigma^2 l_{mx} + \sigma(1-\sigma)(m_y l_{vz} - m_z l_{vy})\right]\mathbf{e}_{235}$ $+\left[\sigma^2 l_{my} + \sigma(1-\sigma)(m_z l_{vx} - m_x l_{vz})\right]\mathbf{e}_{315}$ $+\left[\sigma^2 l_{mz} + \sigma(1-\sigma)(m_x l_{vy} - m_y l_{vx})\right]\mathbf{e}_{125}$
Plane $\mathbf{g}$	$\mathbf{D} \mathbin{\dot\vee} \mathbf{g} \mathbin{\dot\vee} \underset{\sim}{\mathbf{D}} = \sigma g_x \mathbf{e}_{4235} + \sigma g_y \mathbf{e}_{4315} + \sigma g_z \mathbf{e}_{4125}$ $+\left[\sigma^2 g_w - \sigma(1-\sigma)\mathbf{m}\cdot\mathbf{g}_{xyz}\right]\mathbf{e}_{3215}$
Round point $\mathbf{a}$	$\mathbf{D} \mathbin{\dot\vee} \mathbf{a} \mathbin{\dot\vee} \underset{\sim}{\mathbf{D}} = (\sigma a_x + (1-\sigma) m_x a_w)\mathbf{e}_1 + (\sigma a_y + (1-\sigma) m_y a_w)\mathbf{e}_2$ $+(\sigma a_z + (1-\sigma) m_z a_w)\mathbf{e}_3 + a_w \mathbf{e}_4$ $+\left[\sigma^2 a_u + \sigma(1-\sigma)\mathbf{m}\cdot\mathbf{a}_{xyz} + \tfrac{1}{2}(1-\sigma)^2\,\mathbf{m}^2 a_w\right]\mathbf{e}_5$
Dipole $\mathbf{d}$	$\mathbf{D} \mathbin{\dot\vee} \mathbf{d} \mathbin{\dot\vee} \underset{\sim}{\mathbf{D}} = d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + \left[\sigma d_{mx} + (1-\sigma)(m_y d_{vz} - m_z d_{vy})\right]\mathbf{e}_{23}$ $+\left[\sigma d_{my} + (1-\sigma)(m_z d_{vx} - m_x d_{vz})\right]\mathbf{e}_{31} + \left[\sigma d_{mz} + (1-\sigma)(m_x d_{vy} - m_y d_{vx})\right]\mathbf{e}_{12}$ $+\left[\sigma^2 d_{px} + \sigma(1-\sigma)(m_y d_{mz} - m_z d_{my} + m_x d_{pw}) + \tfrac{1}{2}(1-\sigma)^2\left(2m_x\mathbf{m}\cdot\mathbf{d}_{\mathbf{v}} - \mathbf{m}^2 d_{vx}\right)\right]\mathbf{e}_{15}$ $+\left[\sigma^2 d_{py} + \sigma(1-\sigma)(m_z d_{mx} - m_x d_{mz} + m_y d_{pw}) + \tfrac{1}{2}(1-\sigma)^2\left(2m_y\mathbf{m}\cdot\mathbf{d}_{\mathbf{v}} - \mathbf{m}^2 d_{vy}\right)\right]\mathbf{e}_{25}$ $+\left[\sigma^2 d_{py} + \sigma(1-\sigma)(m_x d_{my} - m_y d_{mx} + m_z d_{pw}) + \tfrac{1}{2}(1-\sigma)^2\left(2m_z\mathbf{m}\cdot\mathbf{d}_{\mathbf{v}} - \mathbf{m}^2 d_{vz}\right)\right]\mathbf{e}_{35}$ $+\left[\sigma d_{pw} + (1-\sigma)\mathbf{m}\cdot\mathbf{d}_{\mathbf{v}}\right]\mathbf{e}_{45}$
Circle $\mathbf{c}$	$\mathbf{D} \mathbin{\dot\vee} \mathbf{c} \mathbin{\dot\vee} \underset{\sim}{\mathbf{D}} = c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412}$ $+\left[\sigma c_{gw} - (1-\sigma)\mathbf{m}\cdot\mathbf{c}_{gxyz}\right]\mathbf{e}_{321} + \left[\sigma c_{vx} + (1-\sigma)(m_y c_{gz} - m_z c_{gy})\right]\mathbf{e}_{415}$ $+\left[\sigma c_{vy} + (1-\sigma)(m_z c_{gx} - m_x c_{gz})\right]\mathbf{e}_{425} + \left[\sigma c_{vz} + (1-\sigma)(m_x c_{gy} - m_y c_{gx})\right]\mathbf{e}_{435}$ $+\left[\sigma^2 c_{mx} + \sigma(1-\sigma)(m_y c_{vz} - m_z c_{vy} - m_x c_{gw}) + \tfrac{1}{2}(1-\sigma)^2\left(2m_x\mathbf{m}\cdot\mathbf{c}_{gxyz} - \mathbf{m}^2 c_{gx}\right)\right]\mathbf{e}_{235}$ $+\left[\sigma^2 c_{my} + \sigma(1-\sigma)(m_z c_{vx} - m_x c_{vz} - m_y c_{gw}) + \tfrac{1}{2}(1-\sigma)^2\left(2m_y\mathbf{m}\cdot\mathbf{c}_{gxyz} - \mathbf{m}^2 c_{gy}\right)\right]\mathbf{e}_{315}$ $+\left[\sigma^2 c_{mz} + \sigma(1-\sigma)(m_x c_{vy} - m_y c_{vx} - m_z c_{gw}) + \tfrac{1}{2}(1-\sigma)^2\left(2m_z\mathbf{m}\cdot\mathbf{c}_{gxyz} - \mathbf{m}^2 c_{gz}\right)\right]\mathbf{e}_{125}$
Sphere $\mathbf{s}$	$\mathbf{D} \mathbin{\dot\vee} \mathbf{s} \mathbin{\dot\vee} \underset{\sim}{\mathbf{D}} = s_u\mathbf{e}_{1234} + (\sigma s_x - (1-\sigma) m_x s_u)\mathbf{e}_{4235}$ $+(\sigma s_y - (1-\sigma) m_y s_u)\mathbf{e}_{4315} + (\sigma s_z - (1-\sigma) m_z s_u)\mathbf{e}_{4125}$ $+\left[\sigma^2 s_w - \sigma(1-\sigma)\mathbf{m}\cdot\mathbf{s}_{xyz} + \tfrac{1}{2}(1-\sigma)^2\,\mathbf{m}^2 s_u\right]\mathbf{e}_{3215}$

References

- Projective Geometric Algebra Illuminated
- projectivegeometricalgebra.org





Contact

- lengyel@terathon.com
- Twitter: @EricLengyel
- Bluesky: @ericlengyel.bsky.social
- LinkedIn: www.linkedin.com/in/eric-lengyel